Dis 1D Notes

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RC Circuits

→ We are learning how to solve diff eqns of the form:

$$\boxed{\frac{dx(t)}{dt} + ax(t) = b(t)}$$

2 cases so far

1. $b(t) = 0$ (no input)

$$\frac{dx(t)}{dt} + ax(t) = 0$$

Solutions $x_h(t)$ are known as homogeneous solutions.

2. $b(t) = b = \text{const.}$

$$\frac{dx(t)}{dt} + ax(t) = b$$

Why do we care about $x_h(t)$?

→ Suppose we find any particular

solution $x_p(t)$ that solves it.

$$\frac{dx_p(t)}{dt} + a x_p(t) = b$$ ✓

Not done!

$$\left(\frac{d}{dt} + a\right) x_p(t) = b$$

$$+ \left(\frac{d}{dt} + a\right) x_h(t) = 0 \leftarrow \text{"fancy" } 0$$

$$\left(\frac{d}{dt} + a\right)\left(x_p(t) + x_h(t)\right) = b$$

$x = x_p(t) + x_h(t)$ is also a s/n!

$$\boxed{\begin{gathered} x_g(t) = x_p(t) + C x_h(t) \\ \text{general solutions} \end{gathered}}$$

By linearity.

Familiar? Consider linear algebra

$$A\vec{x} = \vec{b}$$

$$A\vec{x_p} = \vec{b}$$

$$A\vec{x}_{h_1} = \vec{0}$$

$$A\vec{x}_{h_2} = \vec{0}$$

$$\vec{x}_g(t) = \vec{x}_p(t) + \sum_i c_i \vec{x}_{h,i}$$

Deep connection b/w DE and linear algebra

$$\left(\frac{d}{dt} + a\right) \leftarrow \text{linear operator}$$

What are the solutions?

(1) Homogeneous

$$\frac{dx(t)}{dt} = -a\,x(t)$$

Proof 1: Separation of Variables (Calculus)

$$\int \frac{dx}{x} = \int -a\,dt$$

$$\ln x = -at + C$$

$$\boxed{x = C' e^{-at}}$$

$$\boxed{X_h = C' e^{-at}}$$

Proof 2: Guess and check

(~ linear algebra)

linear operator $\dfrac{d\, x_h(t)}{dt} = -a\, x(t)$

$$\frac{d}{dt}\left(X_h\right) = -a X_h$$ eigenvalue

$$\boxed{X_h(t) = C' e^{-at}}$$

In both cases, the solution is <u>not</u> unique!

Reason: C' could be any const

~~Add~~ additional constraint: initial conditions

$$\left.\begin{aligned} X(0) &= X_0 \\ X_h(t) &= C' e^{-at} \\ x_h(0) &= C' e^{0} = C' = X_0 \end{aligned}\right\} \rightarrow \boxed{X_h(t) = X_0 e^{-at}}$$

② Constant Input

Proof 1: Calculus way
(sep. of variables)

$$\frac{dx}{dt} = -ax + b = -a\left(\frac{-b}{a} + x\right)$$

$$\int \frac{dx}{x - \frac{b}{a}} = \int -a\, dt$$

Use a table:

$$\boxed{x(t) = \underbrace{\frac{b}{a}}_{x_p} + \underbrace{C' e^{-at}}_{x_h}}$$

Proof 2: Linear algebra way
(guess and check)

$$\left(\frac{d}{dt} + a\right) x(t) = b$$

$x_p = \frac{b}{a}$ Guess as particular solution

$$\left(\frac{d}{dt}+a\right)\frac{b}{a} = 0 + b = b \checkmark$$

Done? NO! Need homog. sln.

$$x_h(t) = C' e^{-at}$$

$$\boxed{\begin{aligned} x_J(t) &= \frac{b}{a} + C' e^{-at} \\ &= x_p(t) + C' x_h(t) \end{aligned}}$$

Prof 3: Math trick
(change of variables)

$$\frac{dx}{dt} = -ax + b = -a\left(x - \frac{b}{a}\right)$$

Change of vars: $\tilde{x} = x - \frac{b}{a}$

Notice: $$\frac{d\tilde{x}}{dt} = \frac{dx}{dt} - \cancel{\frac{d}{dt}\left(\frac{b}{a}\right)}^{\,0} = \frac{dx}{dt}$$

$dx \quad d\tilde{x}$

$$dt \quad - \quad \overline{dt} = -a\left(x - \frac{b}{a}\right) = -a\tilde{x}$$

eigenvalue or homogeneous eqn

$$\tilde{x}(t) = C' e^{-at}$$

$$x - \frac{b}{a} = C' e^{-at}$$

$$\boxed{x(t) = \frac{b}{a} + C' e^{-at}}$$

Don't forget initial condition!

$$x(0) = x_0$$

$$x(t)\Big|_{t=0} = \frac{b}{a} + C' e^{-at}\Big|_{t=0}$$

$$\Rightarrow x_0 = \frac{b}{a} + C' \longrightarrow C' = x_0 - \frac{b}{a}$$

$$\boxed{x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right) e^{-at}}$$

Nuclear option $\longrightarrow$ plug in values

Why so many methods?

1) Many ways to look at the same problem!

2) Pick whichever makes the most sense to you!

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e^{-at} → pop up all the time in EE (all engineering, physical sciences)

→ transient behavior

e^{at}

Dis 1D Worksheet

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1 RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor over time in an RC circuit. We set up our problem by first defining three functions over time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, and $V_o(t)$ is the voltage across the capacitor at time t.

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_o = \frac{Q}{C}$ where Q is the charge across the capacitor.

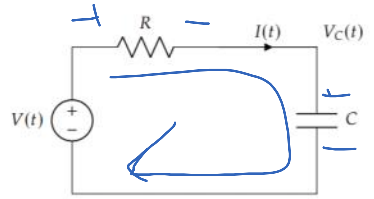


Figure 1: Example Circuit

a) First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.

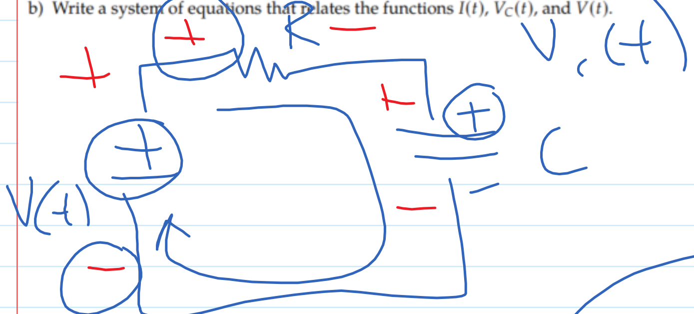
$$i_c(t) = C\frac{dV_c(t)}{dt}$$

derivation

$$\frac{d}{dt}(Q = CV) \longleftarrow \text{capacitor}$$

$$I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

b) Write a system of equations that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.



$I(t)\,R$

$$-V(t) + V_R(t) + V_c(t) = 0$$

$$\boxed{-V(t) + I(t)R + V_c(t) = 0}$$

$$\boxed{-V(t) + I(t)R + V_c(t) = 0}$$

c) Rewrite the previous equation in part (b) in the form of a differential equation involving only $V_C(t)$ and $V(t)$.

$$I(t) = C\frac{dV_c(t)}{dt}$$

$$\boxed{V(t) = RC\frac{dV_c(t)}{dt} + V_c(t)}$$

$$\frac{dx}{dt} = -ax(t) + b(t)$$

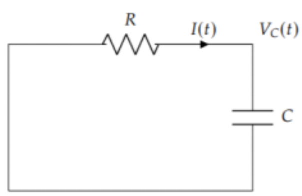


Figure 2: Circuit for part (d)

$$V_c(0) = V_{DD}$$

$$V(t) = 0$$

d) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

$$\frac{dV_c(t)}{dt} = -\frac{1}{RC}V_c(t)$$

$\Longrightarrow$ Guess and check!

$$V_c(t) = V_c(0)e^{-t/RC} \quad \checkmark$$

$$\frac{dV_c}{dt} = V_c(0) \times -\frac{1}{RC}e^{-t/RC} = \frac{V_c(t)}{-RC}$$

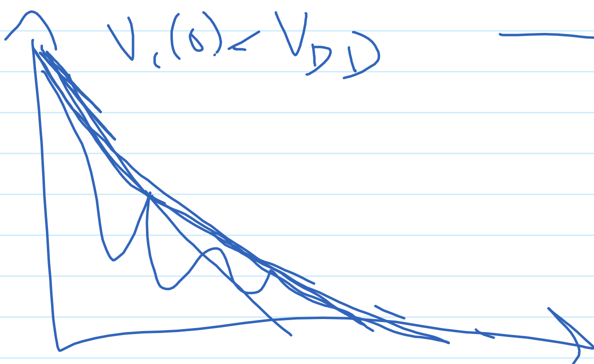
dt RC $-RC$

$$\boxed{V_c(t) = V_{DD}\, e^{-t/RC}}$$

$V_c(t)$ $V_c(0) = V_{DD}$

→ initially, cap is fully charged

→ expect current to flow



$V_c(0) < V_{DD}$ → Cap discharging

$Q = CV$

Expect $V_c(\infty) = 0$

$i = \frac{V_c}{R}$

$t=0,\ V_c(0) = V_{DD},\ i = \frac{V_{DD}}{R}$

$t=\infty,\ V_c(\infty) = 0,\ i = 0$

$i = C\frac{dV_c}{dt}$ → slope of $V_c(t)$

$V_c(0) = 0$

$V(t) = V_{DD}$

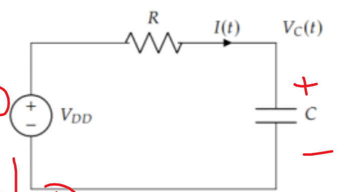
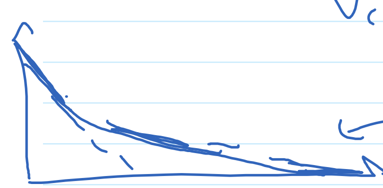


Figure 3: Circuit for part (e)

slope $\sim \frac{V_{DD}}{R}$



slope ~ 0

e) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

In summary,

$V_c \downarrow \rightarrow i \downarrow \rightarrow \frac{dV_c}{dt} \downarrow$

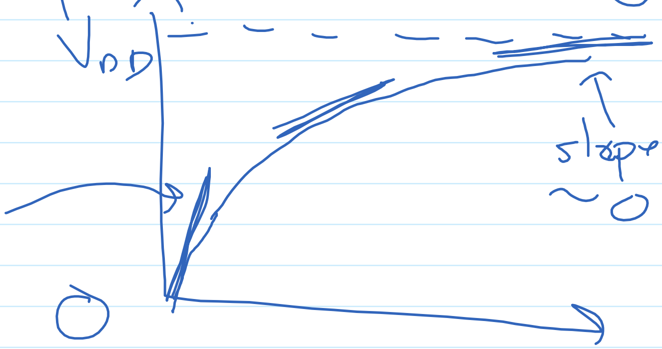
* cap initially → $t = 0$,

* cap initially $\xrightarrow{\text{short}}$ $I(0) = \dfrac{V_{DD}}{R}$

$V_c \downarrow \rightarrow i \downarrow \rightarrow \dfrac{dV_c}{dt} \downarrow$

* cap charged $\longrightarrow t = \infty$, $\xrightarrow{\text{open}}$ $I(\infty) = 0$

$$i \propto \frac{dV_c}{dt}$$



slope $\approx \dfrac{V_{DD}}{R}$

$$V_{DD} = RC\frac{dV_c}{dt} + V_c(t)$$

I. Guess and check

$$x_g(t) = x_p(t) + x_h(t)$$

$$V_c(t) = B + Ae^{-t/RC}$$

$$V_{DD} = RC\frac{d}{dt}\big(\cancel{B} + \underbrace{Ae^{-t/RC}}_{V_c(t)}\big) + V_c(t)$$

$$= \cancel{RC} \times A \times -\frac{1}{\cancel{RC}}e^{-t/RC}$$

$$+ B + Ae^{-t/RC}$$

$$\Longrightarrow B = V_{DD}$$

Initial condition: $V_c(0) = 0$

Initial condition: $V_c(0) = 0$

$$V_c(0) = 0 = V_{DD} + A\cancel{e^{-0/RC}}^{\,1}$$

$$A = -V_{DD}$$

$$\boxed{V_c(t) = V_{DD}\left(1 - e^{-t/RC}\right)}$$

$$V_c(0) = V_{DD}(1-1) = 0 \quad ✓$$

$$V_c(\infty) = V_{DD}(1-0) = V_{DD} \quad ✓$$

* Change of variables

$$RC\frac{dV_c}{dt} + V_c = V_{DD}$$

$$\longrightarrow \quad \frac{dV_c}{dt} = \frac{V_{DD} - V_c(t)}{RC} = -\frac{1}{RC}\left(V_c(t) - V_{DD}\right)$$

$$\frac{d\tilde{V}}{dt} = \frac{dV_c}{dt} \longrightarrow \text{homog. eqn}$$

$$\tilde{V}_c = V_c(t) - V_{DD} \implies \frac{d\tilde{V}_c}{dt} = \frac{dV_c}{dt}$$

$$\frac{d\tilde{V}_c}{dt} = -\frac{1}{RC}\tilde{V}_c \implies \tilde{V}_c(t) = \tilde{V}_c(0)e^{-t/RC}$$

2 Graphing RC Responses

Consider the following RC Circuit with a single resistor R, capacitor C, and voltage source $V(t)$.

$$\tilde{V}_c(0) = V_c(0) - V_{DD}$$
$$= -V_{DD}$$
$$\tilde{V}_c(t) = -V_{DD}e^{-t/RC}$$
$$= V_c(t) - V_{DD}$$
$$\implies \boxed{V_c(t) = V_{DD}\left(1 - e^{-t/RC}\right)}$$

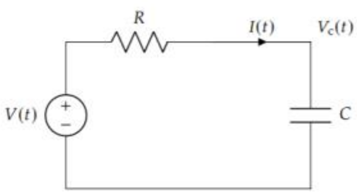
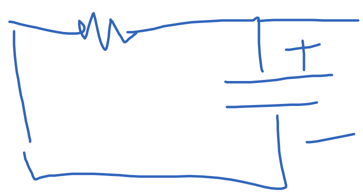
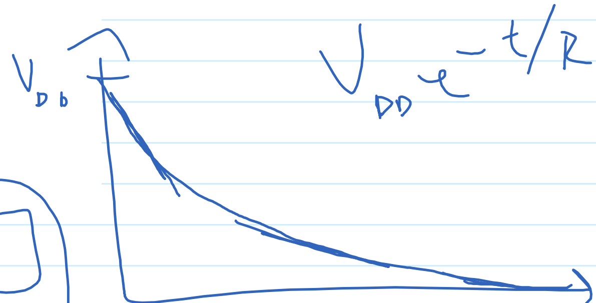


Figure 4: Example Circuit

a) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_c(0) = V_{DD}$) and that $V(t) = 0$ for all $t \geq 0$. Plot the response $V_c(t)$.



$V_c(0) = V_{DD}$



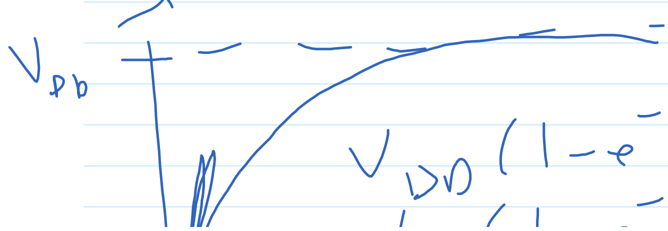
$V_{DD}e^{-t/RC}$

$$\boxed{\tau = RC = [s]}$$

$$V_c(t) = V_{DD}e^{-t/RC} = V_{DD}e^{-t/\tau}$$

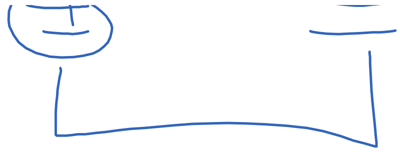
unitless

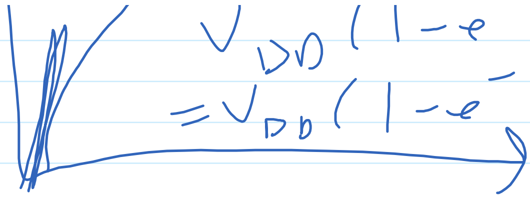
b) Now let's suppose that at $t = 0$, the capacitor is uncharged ($V_c(0) = 0$) and that $V(t) = V_{DD}$ for all $t \geq 0$. Plot the response $V_c(t)$.



$V_{DD}\left(1 - e^{-t/RC}\right)$

$\left(1 - e^{-t/\tau}\right)$





$V_{DD}(1-e^{-t/RC})$

$= V_{DD}(1-e^{-t/\tau})$

To better understand our responses, we now define a **time constant** which is a measure of how long it takes for the capacitor to charge or discharge. Mathematically, we define τ as the time at which $V_C(\tau)$ is $\frac{1}{e} = 36.8\%$ away from its steady state value.

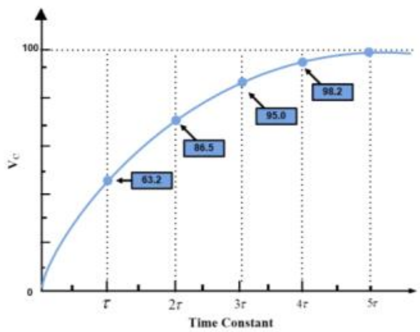


Figure 5: Different values of capacitor voltage at different times, relative to τ.

τ

- useful engineering tool
- measure of how "fast" your ckt is

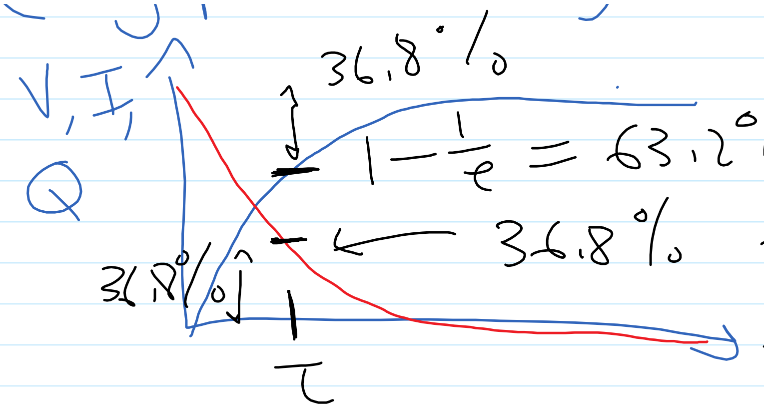
settling time, charge/discharge time

c) Suppose that $V_{DD} = 5\,\text{V}$, $R = 100\,\Omega$, and $C = 10\,\mu\text{F}$. What is the time constant τ for this circuit?

$\tau = RC = 100\,\Omega \times 10\,\mu F$
$= 1000\,\mu s = 1\,ms$

After 1τ, we are $\frac{1}{e} \approx 37\%$ away from our steady-state (asymptotic value)

V_T 36.8%



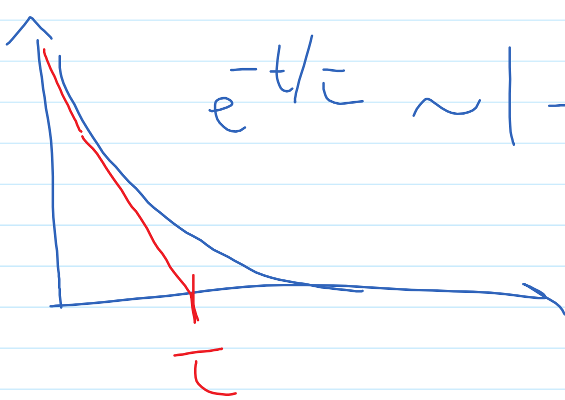
$1 - \frac{1}{e} = 63.2\%$ of final value

← 36.8% of initial value

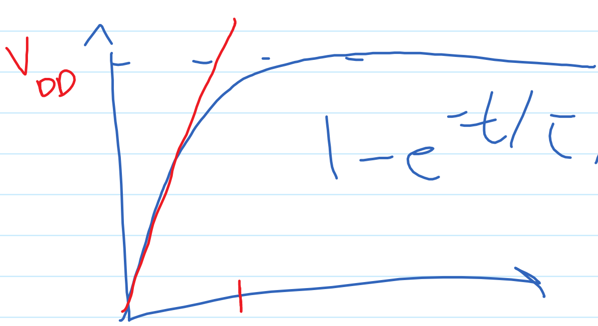
t	τ	3τ	5τ
% from initial value	63%	95%	99%

How long is "long enough"?

$\tau \longrightarrow$ natural timescale



$e^{-t/\tau} \sim 1 - \frac{t}{\tau} \Rightarrow = 0$ when $t = \tau$



$1 - e^{-t/\tau} \sim \frac{t}{\tau} \Rightarrow = V_{DD}$ when $t = \tau$

d) Going back to part (b), on what order of magnitude of time (nanoseconds, milliseconds, 10's of seconds, etc.) does this circuit settle (V_c is > 95% of its value as $t \to \infty$)?

$\tau = 1\,ms \longrightarrow$ order of magnitude is ms

explicitly: $3\tau = 3$ ms

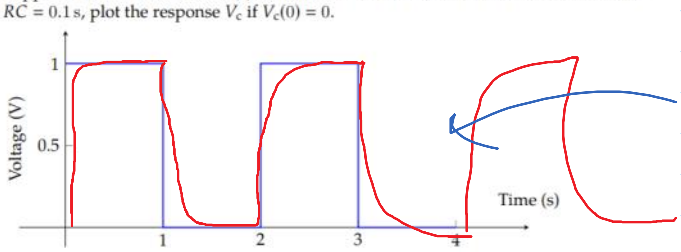
e) Give 2 ways to reduce the settling time of the circuit if we are allowed to change one component in the circuit.

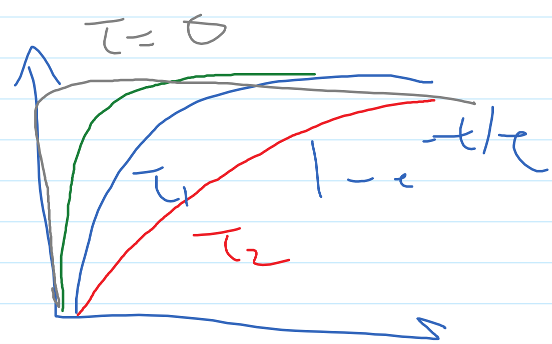
reduce R

reduce C

Which one is better? HW1, Q5

f) Suppose we have a source $V(t)$ that alternates between 0 and $V_{DD} = 1\,V$. Given $RC = 0.1\,s$, plot the response V_c if $V_c(0) = 0$.





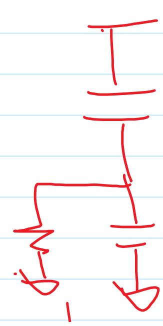
$\tau = 0.1s$

$\Delta T = 1s = 10\tau$

$\tau_2 > \tau_1$?

$\tau_3 < \tau_1$?

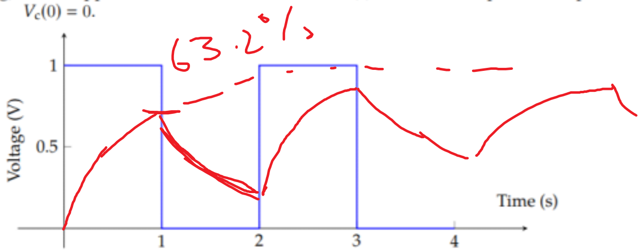
Xstor model:



Careful about τ!

> 1 capacitor

g) Now suppose we have the same source $V(t)$ but $RC = 1\,\mathrm{s}$, plot the response V_c if $V_c(0) = 0$.



36.8%

$\tau = 1s$

$\Delta T = 1s = \tau$

After 2s: $V_{DD}\left(1 - \frac{1}{e}\right)\frac{1}{e} \approx 0.28 V_{DD}$

$$V_c(\tau) = V_{DD}\left(1 - \frac{1}{e}\right)$$