Dis 2B Notes

Tuesday, June 30, 2020 12:02 PM

Phasors

I. The Big Picture (High Level)

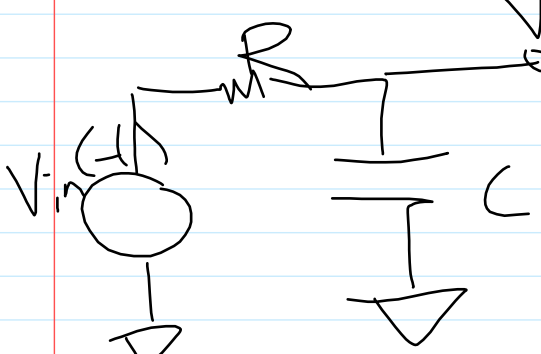
Why have we been learning about diff eq, but jump to complex #s and "phasors"

$\Longrightarrow$ Want to make solving diff eq easier

Recall:

input

$$\boxed{\frac{dx(t)}{dt} + a\,x(t) = b(t)}$$



$$RC\frac{dV_{out}(t)}{dt} + V_{out}(t) = V_{in}(t)$$

Example

$$x(t) = e^{-a(t-t_0)}x_0 + \int_{t}^{t} e^{-a(t-t')}u(t')\,dt'$$

$x(t) = e \quad x_0 + \int_{t_0} e \quad u(t')\,dt'$

$x(t_0) = x_0$

$x_h(t)$ $\qquad$ $x_p(t)$

(t' is a dummy variable)

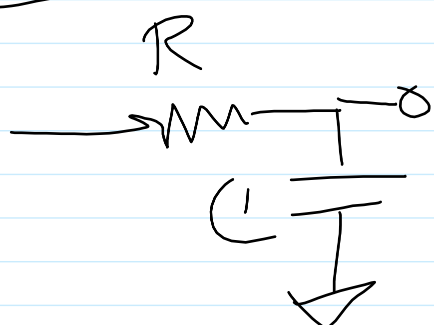
Connection to complex #s:

Plug in $e^{j\omega t}$ (for some reason)

w/ appropriate boundary conditions (e.g. $t \to \infty$ so e^{-at} vanishes)

$\Longrightarrow$ We get a steady state response that is just a scaled version of input.

$V_{in}(t) = \tilde{V}_{in} e^{j\omega t}$



$V_{out}(t) = \tilde{V}_{out} e^{j\omega t}$

$= \lambda \tilde{V}_{in} e^{j\omega t}$

complex numbers

Exploiting eigenfn property of

Exploiting eigenfn property of exponentials under differentiation

Suggest we can skip the integral

$$RC\frac{dV_{out}}{dt} + V_{out} = V_{in}$$

$$V_{out}(t) = \tilde{V}_{out}\, e^{j\omega t}$$

$$V_{in}(t) = \tilde{V}_{in}\, e^{j\omega t}$$

$$RC\left(j\omega \tilde{V}_{out} \cancel{e^{j\omega t}}\right) + \tilde{V}_{out} \cancel{e^{j\omega t}} = \tilde{V}_{in} \cancel{e^{j\omega t}}$$

$$\tilde{V}_{out}\left(1 + j\omega RC\right) = \tilde{V}_{in}$$

$$\tilde{V}_{out} = \lambda \tilde{V}_{in} = \frac{1}{1 + j\omega RC}\tilde{V}_{in}$$

Some Notes

* The gross integral $\int e^{-a(t-t')} u(t')\,dt'$

works for any input, not just $e^{j\omega t}$

* But integrals hard
* What if we wanted to look at a more complicated diff eq (e.g. more complex circuit)

→ take hint of eigenfn property of $e^{j\omega t}$

Expect method of plug in $\tilde{V}_{in} e^{j\omega t}$, get out $\tilde{V}_{out} e^{j\omega t}$ to work for any diff eq only containing derivatives and multiplying by constants

$$\tilde{V}_{in} e^{j\omega t} \longrightarrow \boxed{\begin{array}{l} \text{derivatives } \frac{d^n}{dt^n} \\ \text{scaling by } a_n \end{array}} \longrightarrow \lambda \tilde{V}_{in} e^{j\omega t}$$

But why $e^{j\omega t}$???

The (short) answer:

we care about sinusoidal inputs

(e.g. ~~60~~ Hz ~~AC~~ from wall)

$$\boxed{V_{in}\cos(\omega t+\varphi) = \mathrm{Re}\{\tilde{V}_{in}e^{j\omega t}\}}$$

$\tilde{V}_{in}$ is called the $\boxed{\text{phasor}}$ of the signal $V_{in}(t)$

$$\boxed{\tilde{V}_{in} = \underbrace{V_{in}}_{\text{amplitude of signal}} e^{j\overbrace{\varphi}^{\text{phase of the signal}}}}$$

$$V_{in}\cos(\omega t+\varphi) = \underline{V_{in}e^{j(\omega t+\varphi)} + V_{in}e^{-j(\omega t+\varphi)}}$$

$$= \underline{\overset{1}{(V_{in}e^{j\varphi})e^{j\omega t}} + \overset{2}{(V_{in}e^{-j\varphi})e^{-j\omega t}}}$$

$$= \frac{\underbrace{(V_{in} e^{j\varphi})}_{\tilde{V}_{in}} e^{j\omega t} + \underbrace{(V_{in} e^{-j\varphi})}_{\overline{\tilde{V}_{in}}} e^{-j\omega t}}{2}$$

$$= \frac{\tilde{V}_{in} e^{j\omega t} + \overline{\tilde{V}_{in}} e^{-j\omega t}}{2}$$

Clearly, plugging in $\tilde{V}_{in} e^{j\omega t}$ is easier than $V_{in} \cos(\omega t + \varphi)$

EX:

$$RC \frac{dV_{out}(t)}{dt} + V_{out}(t) = V_{in} \cos(\omega t + \varphi)$$

same equation

$$RC \frac{d}{dt} \operatorname{Re}\{\tilde{V}_{out} e^{j\omega t}\} + \operatorname{Re}\{\tilde{V}_{out} e^{j\omega t}\} = \operatorname{Re}\{\tilde{V}_{in} e^{j\omega t}\} \qquad (1)$$

It can be shown (using linearity mainly) that solving the equation for complex exp.

$$(2)\quad RC\frac{d}{dt}\left(\tilde{V}_{out}e^{j\omega t}\right)+\tilde{V}_{out}e^{j\omega t}=\tilde{V}_{in}e^{j\omega t}$$

also solves the eqn w/ cosine

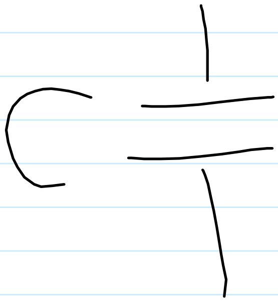
(Solving simpler eqn 2 ⟶ solves 1)

II. The Mechanics
(what you definitely need to know)

Motivation for phasor analysis procedure

* even setting up the diff eq can be hard
* would like an even simpler procedure where we skip the diff eq ⟵ what???

Look at R, L, C :



$$I_c(t) = C\frac{dV_c(t)}{dt}$$

Put in $\tilde{V}_c e^{j\omega t}$ solve for $\tilde{I}_c e^{j\omega t}$

$\downarrow$

Same as if we had put in

$$V_c(t) = \mathrm{Re}\{\tilde{V}_c e^{j\omega t}\} = V_c\cos(\omega t + \varphi)$$

$$\tilde{I}_c \cancel{e^{j\omega t}} = C\frac{d}{dt}\left(\tilde{V}_c e^{j\omega t}\right)$$

$$= Cj\omega \tilde{V}_c \cancel{e^{j\omega t}}$$

$$\boxed{\tilde{V}_c = \frac{1}{j\omega C}\tilde{I}_c = Z_C\tilde{I}_c}$$

Similarly, inductors:

$$V_{in} = \tilde{V}_{in}e^{j\omega t}$$

$$V_l = L\frac{dI_L}{}$$

$$\tilde{V} e^{j\omega t} = L \frac{d}{dt} \tilde{I}_L e^{j\omega t}$$

$$\boxed{\tilde{V}_L = j\omega L \, \tilde{I}_L = Z_L \tilde{I}_L}$$

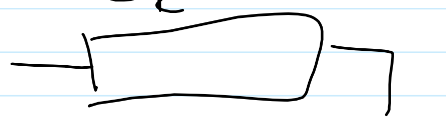
Each ckt element contributes its own diff eq of sorts to the final diff eq (for sinusoidal inputs)

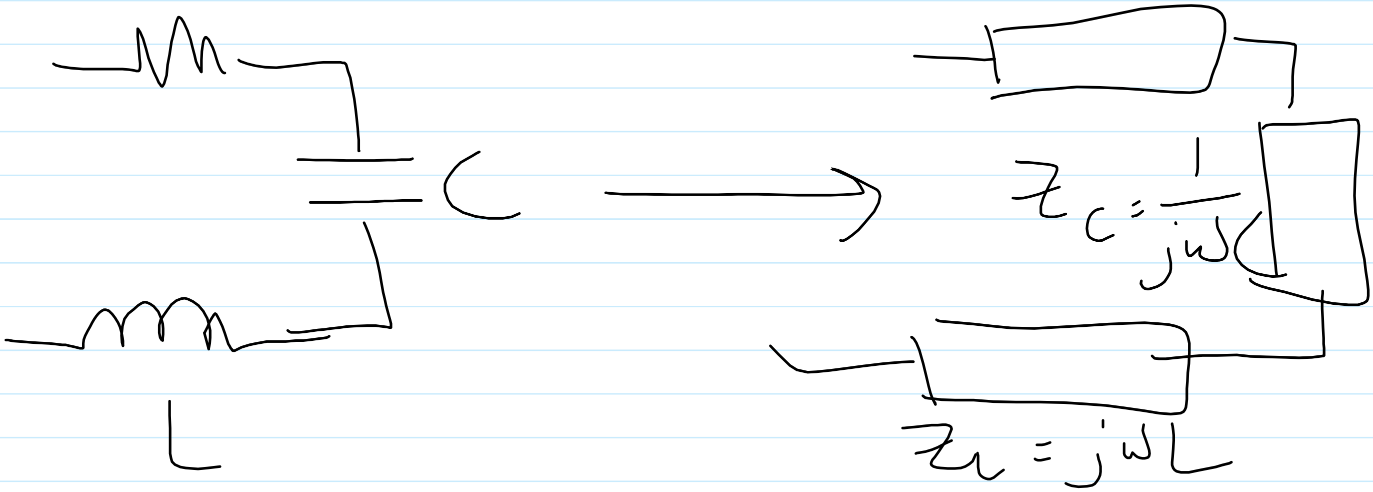
$\Rightarrow$ Then setting up the full diff eq (e.g. RC ckt from earlier) amounts to working with:

* "phasor I-V" $\Rightarrow \tilde{V} = Z\tilde{I}$
* "complex valued resistors"
  $\Rightarrow$ impedances

R

$Z_R = R$





From here: use ckt solving techniques from 16A!

★★★ 5-step method for ★★★ phasor analysis

1. Cosine reference: convert all signals into cosines

$$V_0\cos(\omega t+\phi) = \mathrm{Re}\{\underbrace{V_0 e^{j\phi}}_{\tilde{V}_0 \text{ phasor}} e^{j\omega t}\}$$

2. Transform ckt to phasor domain:

$$V(t) \longrightarrow \tilde{V} = V_0 e^{j\phi_v}$$

$$I(t) \longrightarrow \tilde{I} = I_0 e^{j\phi_i}$$

$$R, L, C \longrightarrow Z_R = R$$

$$Z_L = j\omega L$$

$$Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

$$\frac{1}{j} = -j$$

$$\frac{1}{j} \times \frac{j}{j} = \frac{j}{j^2} = -j$$

(3) Use KCL, KVL in phasor domain.

(4) Solve for unknowns.

(5) Transform back to time domain

$$V_0 e^{j\phi} \longrightarrow V_0 \cos(\omega t + \phi)$$

Dis 2B Worksheet

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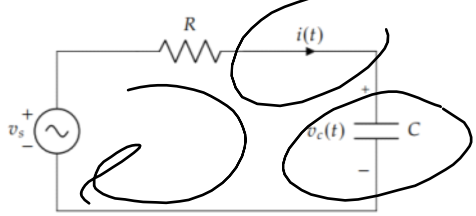
2 Phasor Analysis

Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \Re\mathfrak{e}[Xe^{j\omega t}], \tag{1}$$

where X is a time-independent function called the phasor counterpart of $x(t)$. Thus, $x(t)$ is defined in the time domain, while its counterpart X is defined in the phasor domain.

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{2}$$

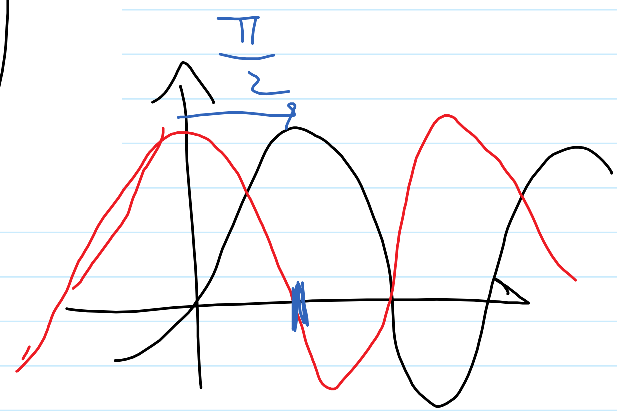
with $\omega = 1 \times 10^3\ \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\,\text{k}\Omega$, and $C = 1\,\mu\text{F}$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source $v_s(t)$.

a) **Step 1: Adopt cosine references**

All voltages and currents with known sinusoidal functions should be expressed in the standard cosine format. Convert $v_s(t)$ into a cosine and write down its phasor representation $\tilde{V}_s$.

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right)$$



convert cos ⟶ sine

shift by $\frac{\pi}{2}$ "forward"

Subtract $\frac{\pi}{2}$

$$v_s(t) = 12\cos\left(\omega t - \frac{\pi}{4} - \frac{\pi}{2}\right)$$

$$= 12\cos\left(\omega t - \frac{3\pi}{4}\right)$$

$$= \mathrm{Re}\left\{V_s e^{j\phi_s} e^{j\omega t}\right\}$$

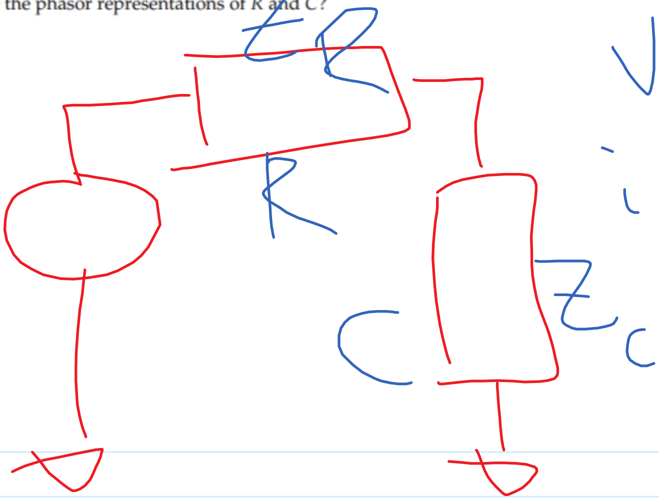
$$\rightarrow \boxed{\tilde{V}_s = 12e^{-j3\pi/4}}$$

b) **Step 2: Transform circuits to phasor domain**

The voltage source is represented by its phasor $\tilde{V}_s$. The current $i(t)$ is related to its phasor counterpart $\tilde{I}$ by

$$i(t) = \Re e[\tilde{I}e^{j\omega t}].$$

What are the phasor representations of R and C?



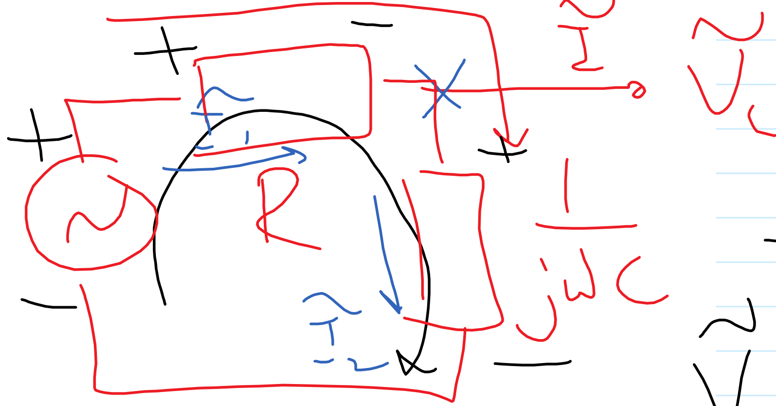
$V_s(t) \longrightarrow \tilde{V}_s$

$i(t) \longrightarrow \tilde{I}$

$Z_R = R$

$Z_C = \dfrac{1}{j\omega C}$

c) **Step 3: Cast KCL and/or KVL equations in phasor domain**
Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.



KCL:

$$\tilde{I}_1 = \tilde{I}_2 = \tilde{I}$$

$$-\tilde{V}_s + \tilde{V}_R + \tilde{V}_c = 0$$

$$\tilde{V}_s = \tilde{I}(Z_R + Z_c)$$

$$\tilde{V}_s = 12e^{-j\frac{3\pi}{4}} = \tilde{I}\left(R + \frac{1}{j\omega C}\right)$$

$$\Rightarrow \tilde{I} = \frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} \times \frac{j\omega C}{j\omega C}$$

$$\boxed{\tilde{I} = \frac{j\omega C\,\tilde{V}_s}{1 + j\omega RC}}$$

$$\tilde{V}_c = Z_c\tilde{I} = \frac{1}{\cancel{j\omega C}} \times \frac{\cancel{j\omega C}\,\tilde{V}_s}{1 + j\omega RC}$$

$$\Rightarrow \tilde{V} = \frac{\tilde{V}_s}{}$$

← Try voltage

$$\Rightarrow \boxed{\tilde{V}_c = \frac{V_s}{1+j\omega RC}}$$

← try voltage divider, see if it matches! (it should)

d) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for $\tilde{I}$ and $\tilde{V}_c$. What are the polar forms of $\tilde{I}$ ($Ae^{j\theta}$, where A is a positive real number) and $\tilde{V}_c$?

$$\tilde{I} = \frac{\tilde{V}_s\, j\omega C}{1+j\omega RC} = \frac{j\, 12 e^{-j\frac{3\pi}{4}} \left(10^3 \frac{\text{rad}}{\text{s}}\right) \times (1\mu F)}{1 + j \left(10^3 \text{ rad/s}\right)(1\mu F)(\sqrt{3}\, k\Omega)}$$

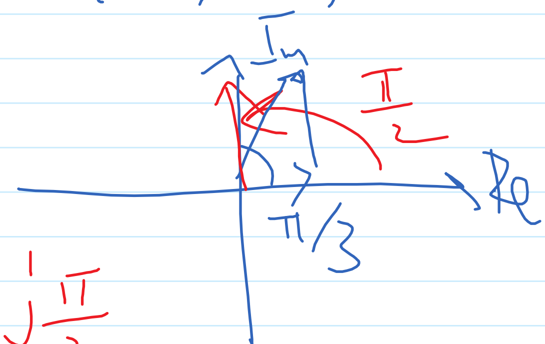
$$= \frac{12 j\, e^{-j\frac{3\pi}{4}} \times 10^{-3} A}{1 + j\sqrt{3}}$$

(polar)

$$= \frac{12 j\, e^{-j\frac{3\pi}{4}}\ mA}{\sqrt{1^2+\sqrt{3}^2}\; e^{j\left(\text{atan2}(\sqrt{3},1)\right)}}$$

$$= \frac{12 j\, e^{-j\frac{3\pi}{4}}\ mA}{2 e^{j\pi/3}}$$

$1 e^{j\frac{\pi}{2}}$



$$= \frac{12 e^{-j\frac{3\pi}{4}} e^{j\frac{\pi}{2}}}{2 e^{j\pi/3}} = \frac{6 e^{-j\frac{\pi}{4}}}{e^{j\pi/3}} \qquad -\frac{\pi}{4} - \frac{\pi}{3} = -\frac{7\pi}{12}$$

$$\boxed{\tilde{I} = 6 e^{-j\frac{7\pi}{\ldots}}}$$

$$\therefore \quad \boxed{\tilde{I} = 6e^{-j\frac{7\pi}{12}}}$$

$$\tilde{V}_c = \frac{\tilde{V}_s}{1+j\omega RC} = \frac{12e^{-j\frac{3\pi}{4}}}{1+j\sqrt{3}} = \frac{12e^{-j\frac{3\pi}{4}}}{2e^{j\pi/3}}$$

$$\boxed{\tilde{V}_c = 6e^{-j\frac{13\pi}{12}}}$$

$$-\frac{3\pi}{4} - \frac{\pi}{3} = -\frac{13}{12}\pi$$

e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_c(t)$? What is the phase difference between $i(t)$ and $v_c(t)$?

$$i(t) = \mathrm{Re}\{\tilde{I}e^{j\omega t}\}$$

$$= \frac{\tilde{I}e^{j\omega t} + \overline{\tilde{I}e^{j\omega t}}}{2} \text{ mA}$$

$$= \frac{6e^{-j\frac{7\pi}{12}}e^{j\omega t} + 6e^{j\frac{7\pi}{12}}e^{-j\omega t}}{2} \text{ mA}$$

$$= \frac{12\cos\left(\omega t - \frac{7\pi}{12}\right)}{2} \text{ mA}$$

$$\boxed{i(t) = 6\cos\left(\omega t - \frac{7\pi}{12}\right) \text{ mA}}$$

$$V_0 \cos(\omega t + \varphi) = \mathrm{Re}\{V_0 e^{j\phi} e^{j\omega t}\}$$

$$\tilde{V}_c = 6e^{-j\frac{13\pi}{12}}$$

$$\Rightarrow \boxed{V_c(t) = 6\cos\left(\omega t - \frac{13\pi}{12}\right) \text{ V}}$$