Dis 2C Notes

Wednesday, July 1, 2020 12:57 PM

# Filters and Transfer Functions

## I. Transfer Functions

$\tilde{V}_{in} e^{j\omega t} \longrightarrow \boxed{\text{ckt}} \longrightarrow \tilde{V}_{out} e^{j\omega t} = \lambda \tilde{V}_{in} e^{j\omega t}$

$\hat{}$ phasor (magnitude + phase information)

$$A\vec{v} = \lambda \vec{v}$$

That value of $\lambda$ is usually important!

$\Longrightarrow$ new name! $\boxed{\text{transfer function}}$

$$\frac{\tilde{V}_{out} e^{j\omega t}}{\tilde{V}_{in} e^{j\omega t}} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \lambda = H(j\omega)$$

$H(\omega)$ doesn't have to be $\dfrac{\tilde{V}_{out}}{\tilde{V}_{in}}$

$\Longrightarrow$ can also be currents $\tilde{I}$

$\dfrac{\tilde{V}_{out}}{\tilde{I}_{in}}$, $\dfrac{\tilde{I}_{out}}{\tilde{I}_{in}}$, $\dfrac{\tilde{I}_{out}}{\tilde{V}_{in}}$

$\dfrac{\tilde{V}}{\tilde{I}} = Z \longleftarrow$ impedance!

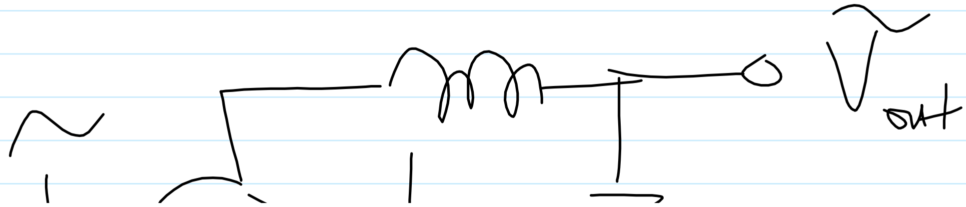
Impedance expression ($Z_L = j\omega L$, $Z_C = \frac{1}{j\omega C}$)

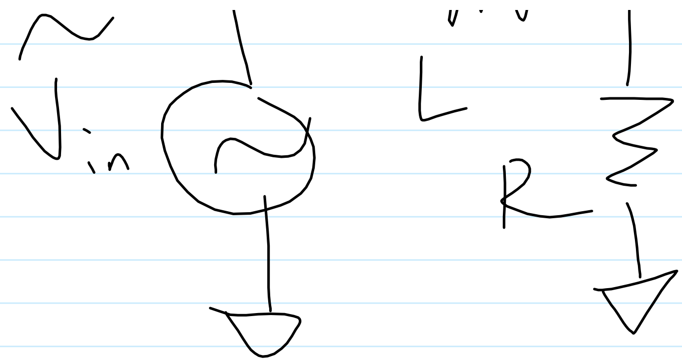
↳ another form of transfer fn

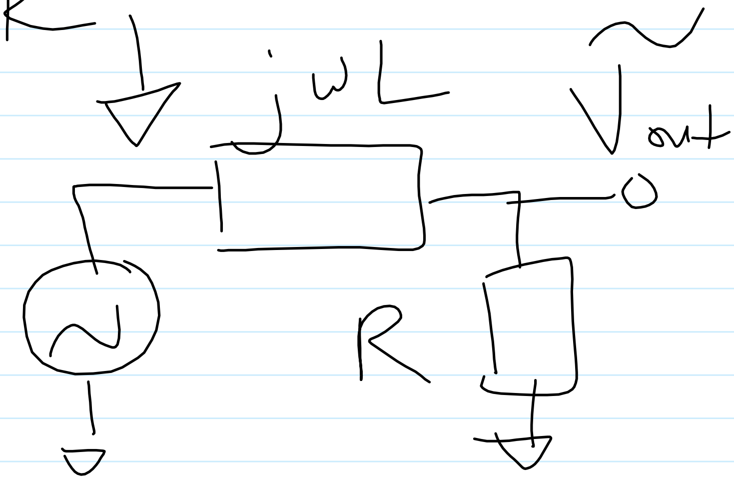
How do you calculate $H(\omega)$?

$\Longrightarrow$ 16A circuit analysis,

resistors $\longrightarrow$ impedances






$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = ?$$

$$\tilde{V}_{out} = \frac{Z_R}{Z_R + Z_L} \tilde{V}_{in}$$

$$\Rightarrow H(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{Z_R}{Z_R + Z_L} = \frac{R}{R + j\omega L}$$

## Recap of Dis 2B

Care about sinusoidal inputs

↳ to simplify analysis,

$$v(t) = V_0 \cos(\omega t + \phi) = \text{Re}\{\underbrace{V_0 e^{j\phi}}_{\text{phasor}} e^{j\omega t}\}$$

$$\boxed{\text{ckt}} \rightarrow \frac{d^n}{dt^n}, \quad a_n$$

$$(j\omega)^n e^{j\omega t} \left(V_0 e^{j\phi}\right)$$

this is a constant relative to $t$

## Some comments on $H(j\omega)$

* complex-valued: has both a magnitude and phase

$$|H(j\omega)| \qquad \angle H(j\omega)$$

* frequency dependent

↳ so you can think of $H(j\omega)$ kind of like a frequency-dependent gain

$\Rightarrow$ recall in 16A:

$$A = \frac{V_{out}}{V_{in}} \quad \text{for an op-amp}$$

Sinusoidal input:

$\Longrightarrow \cos(\omega t)$ or $\sin(\omega t)$

or $\cos(\omega_1 t) + \cos(\omega_2 t) + \dots$

$$\cos(\omega t) = \mathrm{Re}\{e^{j\omega t}\}$$

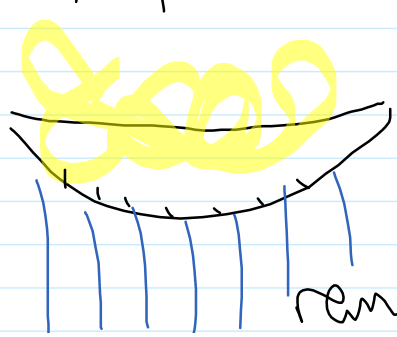
*** Using $e^{j\omega t}$ instead of $\mathrm{Re}\{e^{j\omega t}\} = \cos(\omega t)$ makes the math easier

$$V_0 e^{j\omega t} \longrightarrow V_0 \cos(\omega t + \phi)$$

corresponds to

## II. Filters

pasta

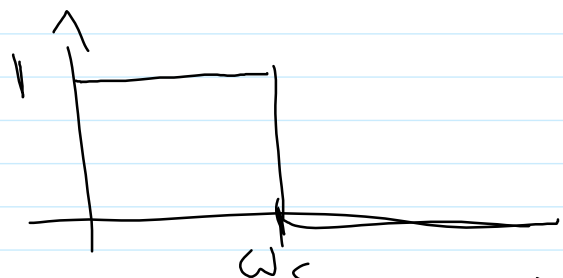


: given a signal, we would like to remove certain parts

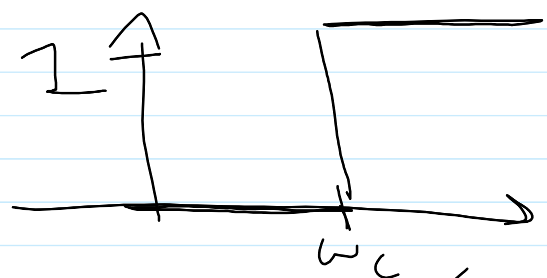
(e.g. high freq. noise)

$\Rightarrow$ part of bigger field called signal processing

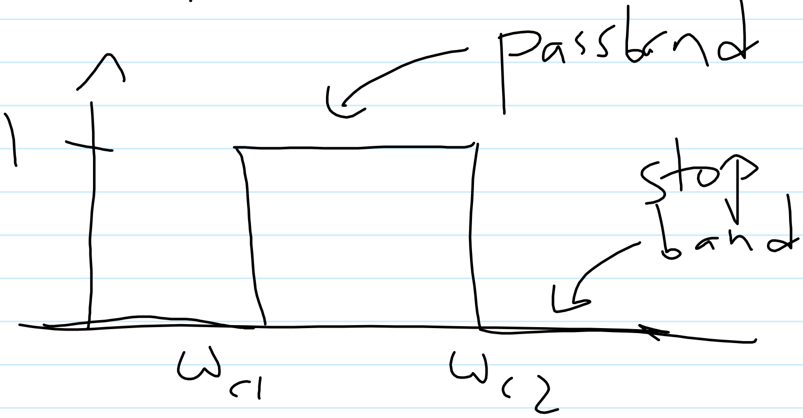
## Ideal Filters



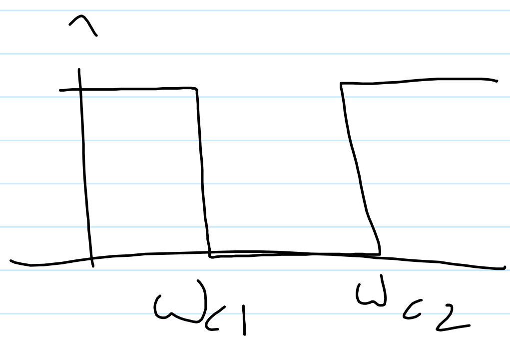

low pass (LPF)




high pass (HPF)



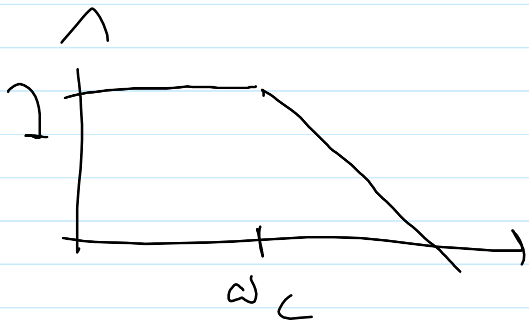

bandpass




bandstop

In practice, never going to get sharp edges

→ same rolloff



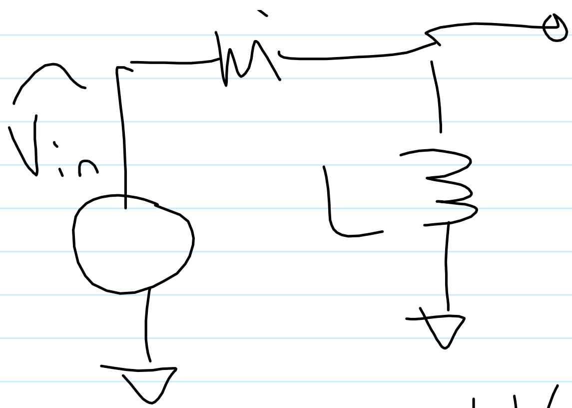




How do we actually implement this?

- digital filtering (software)
  → take EE123
- analog filtering
  → circuits → Transfer fn!

$H(j\omega)$ does some freq-dependent scaling of input signal




$$H(j\omega) = \frac{j\omega L}{j\omega L + R}$$

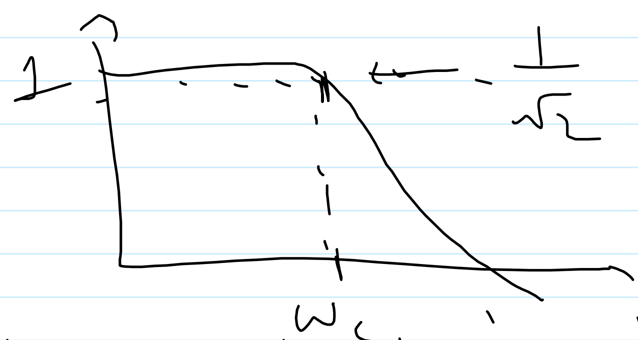
stop LF $\rightarrow$ $H(j0) = \frac{0}{0+R} = 0$

keep HF $H(j\infty) = \frac{j\omega L}{j\omega L} = 1$

$\Longrightarrow$ high pass filter

* Cutoff/Corner Frequency

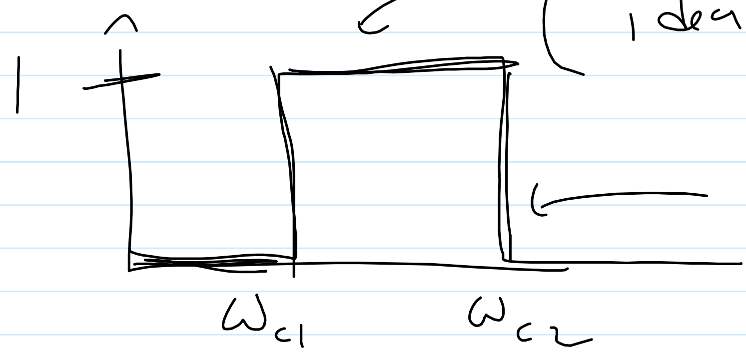
- common definition:
  when the $|H(j\omega)|$ is $\frac{1}{\sqrt{2}}$ of the value at the passband



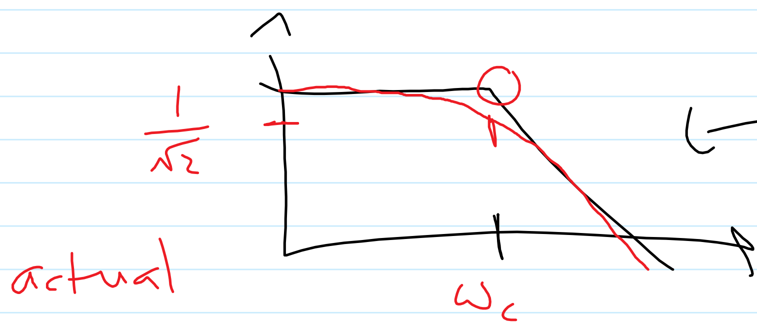

- intuitive definition: when there's

a "cutoff", i.e, a boundary b/w passband and stopband

(ideal BP)




$P = \frac{V^2}{R}$

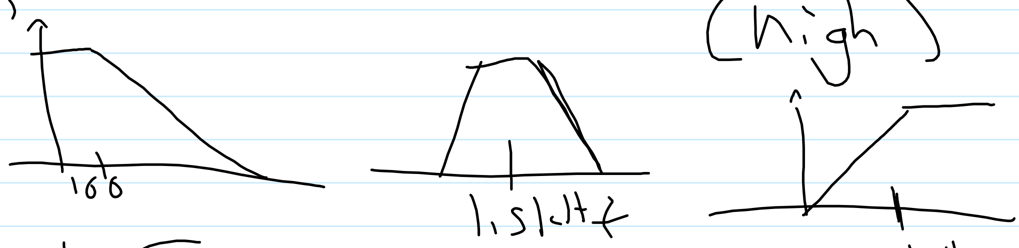



straight line approx

EX: Math's amp

100 Hz 1.5 kHz 12kHz

low mid treble (high)




## General Form of $H(j\omega)$

$$H(j\omega) = \frac{Z(\omega)}{P(\omega)} = K \frac{(j\omega)^{N_{z0}}\left(1 + \frac{j\omega}{\omega_{z1}}\right)\cdots}{(j\omega)^{N_{p0}}\left(1 + j\omega \right)}$$

$p(\omega)$

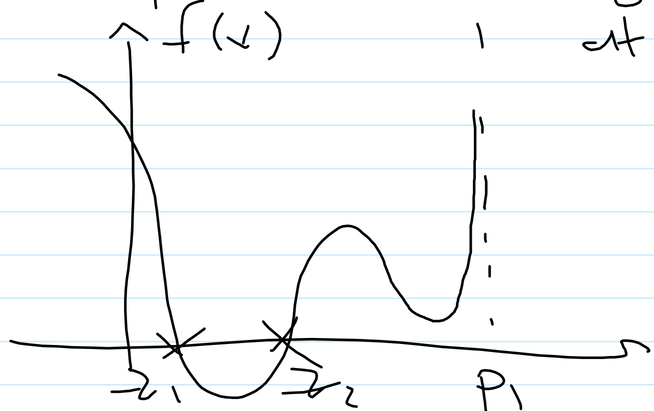
$$\frac{\cdots}{(j\omega)^{N_{p0}}\left(1+\frac{j\omega}{\omega_{p1}}\right)\cdots}$$

pole at origin

zero at origin

$\omega_z$ = "zeroes"

$\omega_p$ = "poles"




zeroes of $f(x)$ are values $z$ s.t.

$$f(z) = 0$$

poles of $f(x)$ are values $p$ s.t.

$$f(p) = \infty$$

$$H(j\omega) = \frac{\left(1+\frac{j\omega}{\omega_{z1}}\right)\left(1+\frac{j\omega}{\omega_{z2}}\right)\cdots}{\left(1+\frac{j\omega}{\omega_{p1}}\right)\left(1+\frac{j\omega}{\omega_{p2}}\right)\cdots}$$

$$s = j\omega$$

$$= \frac{\left(1+\frac{s}{\omega_{z1}}\right)\cdots}{\left(1+\frac{s}{\cdot}\right)\cdots}$$

$$\overline{\left(1+\frac{s}{\omega_{p1}}\right)\cdots}$$

Then, we see that $s = -\omega_{p1}$ is what sends $H(j\omega)$ to $\infty$

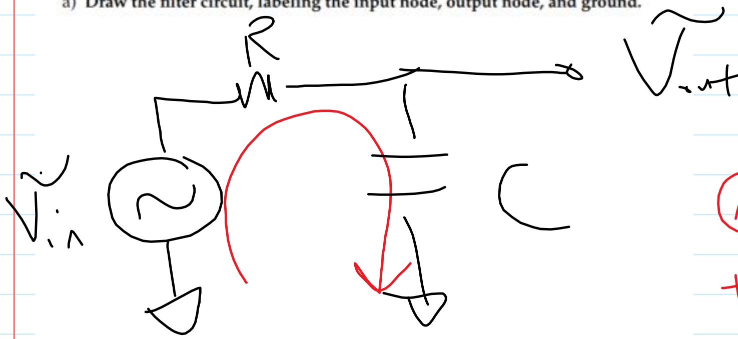
and, $s = -\omega_{z1}$ is what sends $H(j\omega)$ to $0$

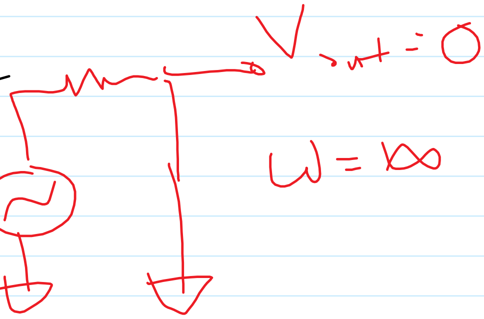
Dis 2C Worksheet

Wednesday, July 1, 2020 12:57 PM

You have a 1 kΩ resistor and a 1 μF capacitor wired up as a low-pass filter.

a) **Draw the filter circuit, labeling the input node, output node, and ground.**

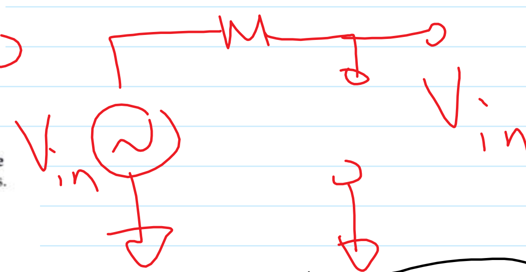



$$Z_C(\omega) = \frac{1}{j\omega C} \quad : \quad \begin{array}{l} \omega \to 0 : \text{open} \\ \omega \to \infty : \text{short} \end{array}$$

$V = ZI$

$\omega = 0$




b) **Write down the transfer function of the filter, $H(j\omega)$ that relates the output voltage phasor to the input voltage phasor.** Be sure to use the given values for the components.

$$H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \boxed{\frac{1}{1 + j\frac{\omega}{10^3}}} = \frac{1}{\left(1 + j\frac{\omega}{\omega_p}\right)}$$

$RC = 1\ k\Omega \times 1\ \mu F$

$F\Omega \quad s^{-3}$

$\omega_p = 10^3$ rad/s

c) **Write an exact expression for the *magnitude* of $H(j\omega = j10^2)$, and give an approximate numerical answer.**

$$|H(j\omega)| = \left| \frac{1}{1 + j\frac{\omega}{\omega_p}} \right|$$

$z = x + jy$

$|z| = \sqrt{x^2 + y^2}$

$$= \frac{1}{\sqrt{1^2 + \left(\frac{\omega}{\omega_p}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{10^2}{10^3}\right)^2}}$$

$$\sqrt{1^2 + \left(\frac{\omega}{\omega_p}\right)^2} \qquad \sqrt{1 + \left(\frac{10^2}{10^3}\right)^2}$$

$$\boxed{|H(j10^2)| = \frac{1}{\sqrt{1 + 10^{-2}}}}$$

$$|z_1 z_2| = |z_1||z_2|$$

$$\boxed{|H(j10^2)| \approx 1} \longleftarrow$$

Specifically, for $\omega << \omega_p$, $\left(\frac{\omega}{\omega_p}\right)^2 << 1$

$$\Longrightarrow |H(j\omega << j\omega_p)| \approx \frac{1}{\sqrt{1}} = 1$$

d) **Write an exact expression for the *magnitude* of $H(j\omega = j10^6)$, and give an approximate numerical answer.**

$$|H(j10^6)| = \frac{1}{\sqrt{1 + \left(\frac{10^6}{10^3}\right)^2}} \boxed{= \frac{1}{\sqrt{1 + 10^6}}}$$

$$|H(j10^6)| \approx \frac{1}{\sqrt{10^6}} = \frac{1}{10^3} \boxed{= 10^{-3}}$$

Specifically, for $\omega >> \omega_p$, $\left(\frac{\omega}{\omega_p}\right)^2 >> 1$

e) **Write an exact expression for the *phase* of $H(j\omega = j1)$, and give an approximate numerical answer.**

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_p}}$$

$$\Longrightarrow |H| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_p}\right)^2}} \approx \frac{1}{\sqrt{\left(\frac{\omega}{\omega_p}\right)^2}} = \frac{\omega_p}{\omega} \quad \star$$

$$\angle\left(\frac{z_1}{z_2}\right) = \angle z_1 - \angle z_2$$

$$\angle \frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \angle e^{j(\theta_1 - \theta_2)} = \theta_1 - \theta_2 = \angle z_1 - \angle z_2$$

$|z_2| \neq 0$ $\qquad = \measuredangle z_1 - \measuredangle z_2$

$$\measuredangle z_1 z_2 = \measuredangle z_1 + \measuredangle z_2 \qquad z = x + jy$$

$$\measuredangle \frac{1}{1 + j\frac{\omega}{\omega_p}} = \cancel{\measuredangle 1}_{0} - \measuredangle\left(1 + j\frac{\omega}{\omega_p}\right)$$

$$= -\measuredangle\left(1 + j\frac{\omega}{\omega_p}\right) = \boxed{-\operatorname{atan2}\left(\frac{\omega}{\omega_p}, 1\right)}$$

$$\measuredangle H(j\omega) = -\operatorname{atan2}\left(\frac{1}{10^3}, 1\right)$$

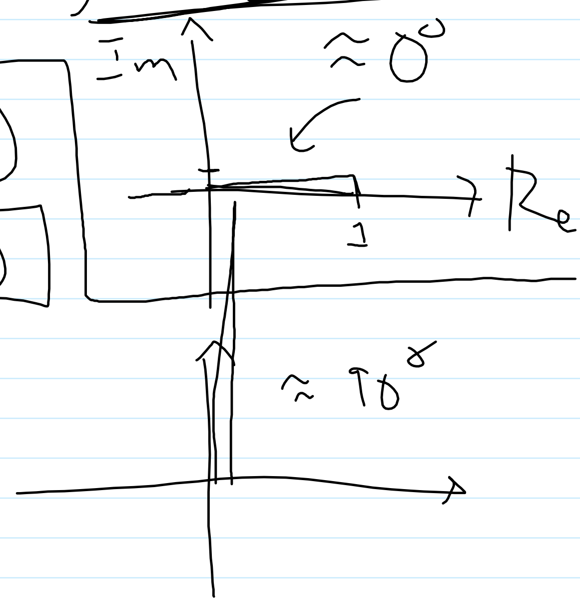
$$= -\arctan(10^{-3}) = -10^{-3}$$

f) **Write an exact expression for the *phase* of $H(j\omega = j10^6)$, and give an approximate numerical answer.**

$$\measuredangle H(j\omega) = -\operatorname{atan2}\left(\frac{10^6}{10^3}, 1\right)$$

$$= \boxed{-\operatorname{atan2}(10^3, 1)}$$

$$\approx \boxed{-90^\circ}$$

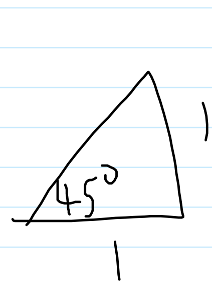



g) **Write down an expression for the *corner frequency* $\omega_c$ of this circuit. Evaluate the *magnitude* and *phase* of $H(j\omega = j\omega_c)$.**

$$\frac{1}{\sqrt{2}} = H(j\omega_c) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_p}\right)^2}} \leftarrow 1$$

$$\Rightarrow \boxed{\omega_c = \omega_p = 10^3 \text{ rad/s}}$$

$$\boxed{|H(j\omega_c)| = \frac{1}{\sqrt{2}}}$$




$$\measuredangle H(j\omega_c) = -\operatorname{atan2}\left(\frac{\omega_c}{\omega_c}, 1\right)$$

$$= -\frac{\pi}{4} = -45°$$

h) **Write down an expression for the time-domain output waveform $V_{out}(t)$ of this filter if the input voltage is $V(t) = 1\sin(1000t)$ V.** You can assume that any transients have died out — we are interested in the steady-state waveform.

$\omega = 10^3$ rad/s $= \omega_c$

$$V(t) = 1\sin(1000t)$$

$$V_{out}(t) = 1\cos\left(1000t - \frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(e^{j\omega t} e^{-j\frac{\pi}{2}} + e^{-j\omega t} e^{j\frac{\pi}{2}}\right)$$

By convention, $\tilde{V}_o = V_o e^{j\phi}$ where $\mathrm{Re}\{\tilde{V}_o e^{j\omega t}\} = V_o \cos(\omega t + \phi)$

$\rightarrow$ ckt $\rightarrow \lambda \tilde{V}_{in} e^{j\omega t} = H(j\omega)\tilde{V}_{in} e^{j\omega t}$

$$H(j\omega) = \frac{\tilde{V}_{out} e^{j\omega t}}{\tilde{V}_{in} e^{j\omega t}} \Longrightarrow H(j\omega_c) = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$$

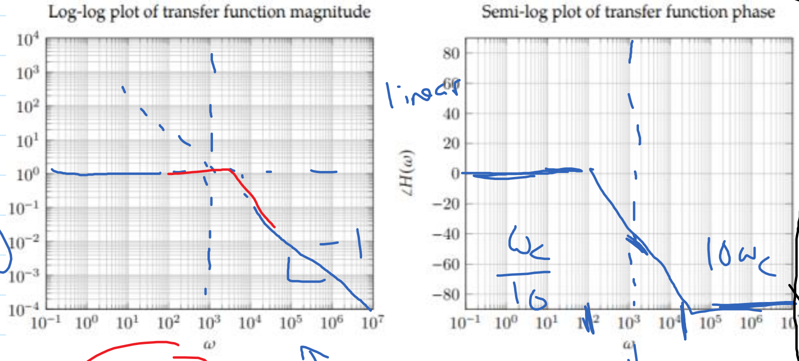
$$H(j\omega_c)\tilde{V}_{in} = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}} e^{-j\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}} e^{-j\frac{3\pi}{4}} = \tilde{V}_{out}$$

convert back to time domain

$$V_{out}(t) = \frac{1}{\sqrt{2}}\cos\left(\omega t - \frac{3\pi}{4}\right)$$

i) **Based off the values calculated in the previous parts, predict what the Bode plots (both magnitude and phase) of the filter will look like and sketch them on the graph paper below. You may use a straight line approximation.**




j) **Use a computer to draw the Bode plots (both magnitude and phase) of the filter. Compare them to your sketch from before.**

j) Use a computer to draw the Bode plots (both magnitude and phase) of the filter. Compare them to your sketch from before.

log

log

$$= \frac{1}{\sqrt{2}} \sin\left(\omega t - \frac{\pi}{4}\right)$$

$\sqrt{2}$

$\omega \ll \omega_c$: $|H| \approx 1$

$\omega \gg \omega_c$: $|H| \approx \dfrac{\omega_c}{\omega}$

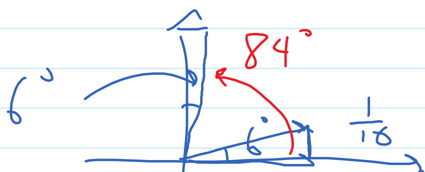
$\omega \ll \omega_c$: $\measuredangle H \approx 0$

$\omega \gg \omega_c$: $\measuredangle H \approx -90^\circ$ $\quad \measuredangle H(\omega_c) = -45^\circ$

$$\omega = \frac{\omega_c}{10}: \quad \measuredangle H\left(j\frac{\omega_c}{10}\right) = -\mathrm{atan2}\left(\frac{\frac{\omega_c}{10}}{\omega_c}, 1\right)$$

$$= -\arctan\left(\frac{1}{10}\right) \approx -0.1 \text{ rad}$$

$$\approx -6^\circ$$



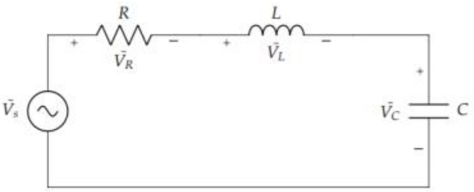

$$\omega = 10\omega_c: \quad \measuredangle H(j10\omega_c) = -\arctan(10) = -84^\circ$$

$$\omega = 2\pi f \longrightarrow f = \frac{\omega}{2\pi}$$

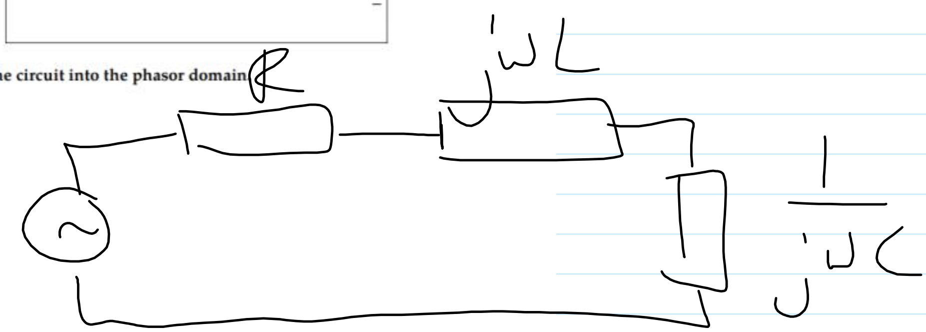
"$2\pi = 10$"

## 3 RLC Circuit

In this question, we will take a look at an electrical systems described by second-order differential equations and analyze it in the phasor domain. Consider the circuit below where $\vec{V}_s$ is a sinusoidal signal, $L = 1\,\text{mH}$, and $C = 1\,\text{nF}$:




a) **Transform the circuit into the phasor domain.**




b) **Solve for the transfer function $H_C(\omega) = \frac{\vec{V}_C}{\vec{V}_s}$ in terms of $R$, $L$, and $C$.**

$$H_c(\omega) = \frac{Z_C}{Z_R + Z_L + Z_C} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

$$= \boxed{\frac{1}{1 + j\omega RC + (j\omega)^2 LC}}$$

c) **Solve for the transfer function $H_L(\omega) = \frac{\tilde{V}_L}{\tilde{V}_s}$ in terms of $R$, $L$, and $C$.**

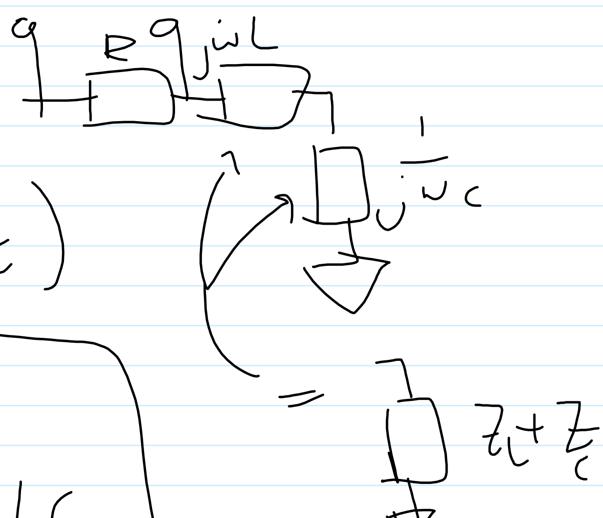
$$H_L(\omega) = \frac{Z_L}{Z_R + Z_L + Z_C}$$

$$= \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}} \times \frac{j\omega C}{j\omega C}$$

$$= \boxed{\frac{(j\omega)^2 LC}{1 + j\omega RC + (j\omega)^2 LC}}$$

d) **Solve for the transfer function $H_R(\omega) = \frac{\tilde{V}_R}{\tilde{V}_s}$ in terms of $R$, $L$, and $C$.**

$$H_R(\omega) = \frac{R}{R + \left(j\omega L + \frac{1}{j\omega C}\right)}$$




$$\boxed{H_R(j\omega) = \frac{j\omega RC}{1 + j\omega RC + (j\omega)^2 LC}}$$

$$\boxed{H_R(j\omega) \quad 1 + j\omega RC + (j\omega)^2 LC}$$

e) **Use a computer to draw the magnitude Bode plots of $H_C(\omega)$, $H_L(\omega)$, and $H_R(\omega)$ when $R = 2\,\mathrm{k}\Omega$.**

$R = 2\,k\Omega, \; L = 1\,mH, \; C = 1\,nF$

$LC = 10^{-12}\,[s^2]$

$RC = 2\times10^{-6}\,[s]$

I. $$H_R(\omega) = \frac{j\omega(2\times10^{-6})}{1 + j\omega(2\times10^{-6}) + (j\omega)^2(10^{-12})}$$

Look at how $H(\omega)$ approaches limits.

$\omega \to 0$

$\omega \to \infty$

$$\lim_{\omega\to0} \frac{j\omega RC}{1 + j\omega RC + (j\omega)^2 LC} = \frac{j\omega RC}{1+0+0^2} = j\omega RC$$

$\omega \to 0 \quad 1 + j\omega RC + (j\omega)^2 LC \quad 1+0+0$

$= j\omega\,(2\times10^{-6})$

$$\lim_{\omega\to\infty} \frac{j\omega RC}{1+j\omega RC+(j\omega)^2 LC} = \frac{\cancel{j\omega} R\cancel{C}}{(j\omega)^{\cancel{2}} L\cancel{C}}$$

$(j\omega)^2 LC$ dominates

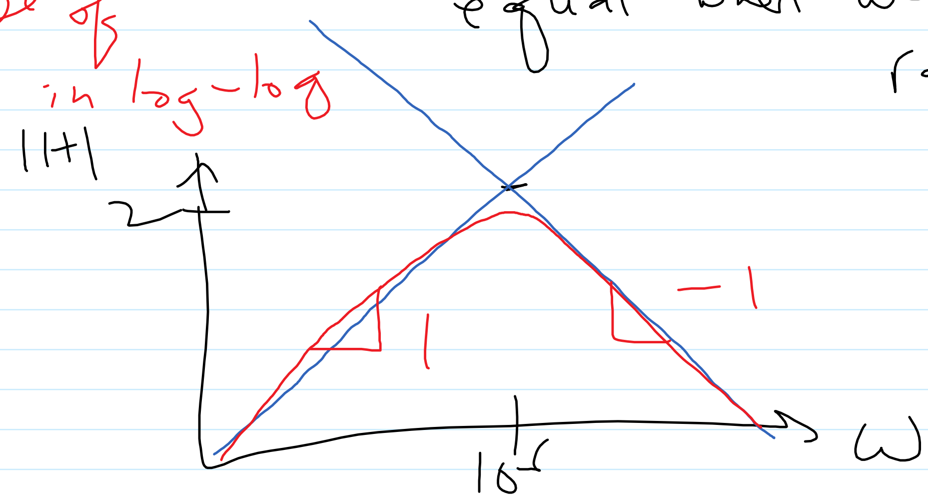
$$= \frac{R}{L}\,\frac{1}{j\omega} = \frac{2\,k\Omega}{1\,mH}\,\frac{1}{j\omega} = \frac{2\times10^{6}}{j\omega}$$

$|H(j\omega\to 0)| = \omega \times (2\times10^{-6})$ ← slope of 1 in log-log

$|H(j\omega\to\infty)| = \dfrac{2\times10^{6}}{\omega}$

Slope of $-1$ in log-log

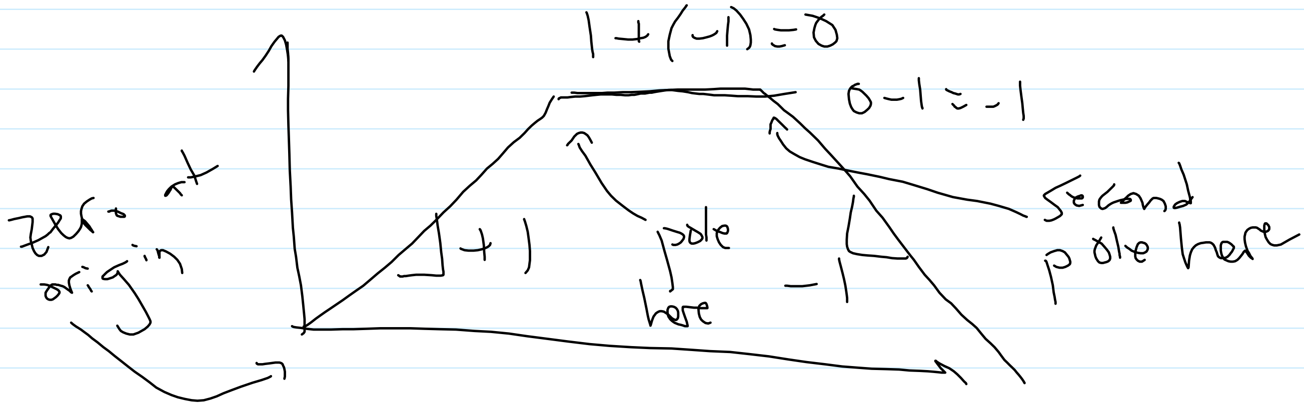
equal when $\omega = 10^6$ rad/s




Tomorrow: pole ⟶ $-1$ in slope

zero ⟶ $+1$ in slope

$1+(-1)=0$




II. $H_L(j\omega)$

$$\Longrightarrow H_L(j\omega) = \frac{(j\omega)^2 LC}{1 + j\omega RC + (j\omega)^2 LC}$$

$$\lim_{\omega \to 0} H_L(\omega) = \frac{(j\omega)^2 LC}{1 + 0 + 0^2} = (j\omega)^2 LC$$

$$= -\omega^2 LC$$

$$= -10^{-12}\omega^2$$

slope 2 in log-log

$$\lim_{\omega \to \infty} H_L(\omega) = \frac{(j\omega)^2 LC}{(j\omega)^2 LC} = 1$$

$10^6$

$10^{-12}\omega^2 = 1$ when $\omega = 10^6$ rad/s

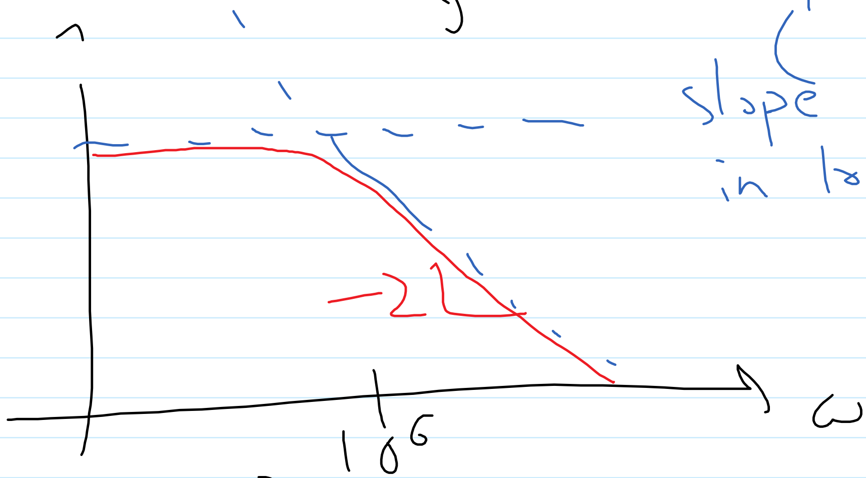
III. $H_C(\omega)$

$$\Longrightarrow H_C(\omega) = \frac{1}{1 + j\omega RC + (j\omega)^2 LC}$$

$$\lim_{\omega \to 0} H_C(\omega) = \frac{1}{1 + 0 + 0} = 1$$

$$\lim_{\omega \to \infty} H_C(\omega) = \frac{1}{(j\omega)^2 LC} = \frac{-10^{12}}{\omega^2}$$

slope −2 in log-log




$10^6$

$\frac{10^{12}}{\omega^2} = 1$ when $\omega = 10^6$ rad/s