Dis 2D Notes

Wednesday, July 1, 2020 4:36 PM

Bode Plots

- ✱ Transfer Function Recap
- ✱ Bode plots: Intro
- ✱ Drawing Bode Plots
- ✱ Reading Bode plots

I. Transfer Functions

$$H(\omega) = K\frac{(j\omega)^{N_{z0}}\left(1+j\frac{\omega}{\omega_{z1}}\right)\left(1+j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1+j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}}\left(1+j\frac{\omega}{\omega_{p1}}\right)\left(1+j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1+j\frac{\omega}{\omega_{pm}}\right)}$$

zeroes

factored form

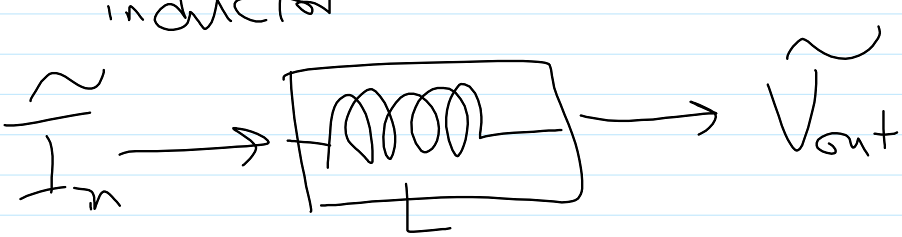
poles

general form

Transfer functions capture $\boxed{\text{input-output behavior}}$

$$H(j\omega) = \frac{\widetilde{out}}{\widetilde{in}}$$

Ex: inductor

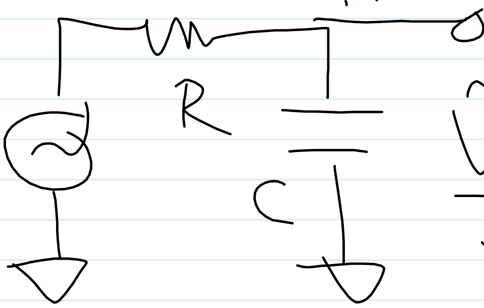


$$\tilde{V}_{out} = Z_L \tilde{I}_{in} = j\omega L \tilde{I}_{in}$$

out

$$Z_L(j\omega) = \frac{\tilde{V}_{out}}{\tilde{I}_{in}} = j\omega L$$

EX2:



$$\frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{Z_C}{Z_C + Z_R}$$

$$H(j\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} \times \frac{j\omega C}{j\omega C} = \frac{1}{1 + j\omega RC}$$

$$= \frac{1}{1 + j\frac{\omega}{\underbrace{\frac{1}{RC}}_{= \omega_p}}}$$

~ From transfer fns to Bode plots! ~

$H(j\omega)$ is a complex number

$\Longrightarrow$ has magnitude $|H(j\omega)|$

and phase $\measuredangle H(j\omega)$

$$H(j\omega) = |H(j\omega)| e^{j \measuredangle H(j\omega)}$$

$\Longrightarrow$ helpful to plot $|H(j\omega)|$ and

$\measuredangle H(j\omega)$ as fns of ω

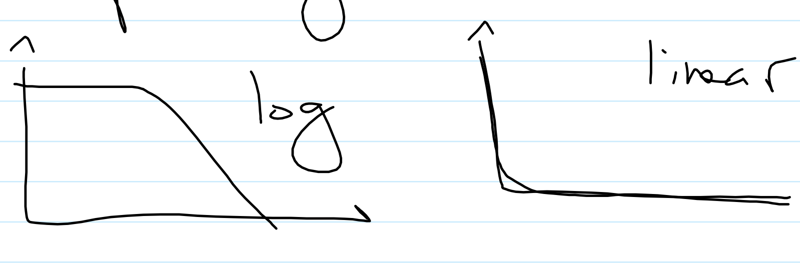
→ Bode plot

II. Bode Plot Intro

1) log-log for magnitude $|H(j\omega)|$
2) log-linear scale for phase $\measuredangle H(j\omega)$

Why?

- changes in $|H|$, $\measuredangle H$ happen over huge range of ω (orders of magnitude)
- Magnitude also changes over orders of magnitude



phase → doesn't change so much
($LPF: 0° \longrightarrow -90°$)
⇒ log-linear

* log scale makes plotting easier!

$$H(j\omega) = \frac{H_{z1}(j\omega) H_{z2}(j\omega) \cdots}{H_{p1}(j\omega) H_{p2}(j\omega) \cdots}$$

$$|z_1 z_2| = |z_1||z_2|$$

$$|H(j\omega)| = \frac{|H_{z1}(j\omega)| \, |H_{z2}(j\omega)| \cdots}{|H_{p1}(j\omega)| \, |H_{p2}(j\omega)| \cdots}$$

Annoying to plot!

But in log scale ...

$$\log AB = \log A + \log B$$

$$\log (A/B) = \log A - \log B$$

$$\log |H(j\omega)| = \log |H_{z1}(j\omega)| + \log |H_{z2}(j\omega)| + \cdots$$
$$- \log |H_{p1}(j\omega)| - \log |H_{p2}(j\omega)| - \cdots$$

We can break up transfer fn into smaller constituent parts

*** This is powerful, because ***

we can smaller constituent plots we already know to get final plot

* What about phase?

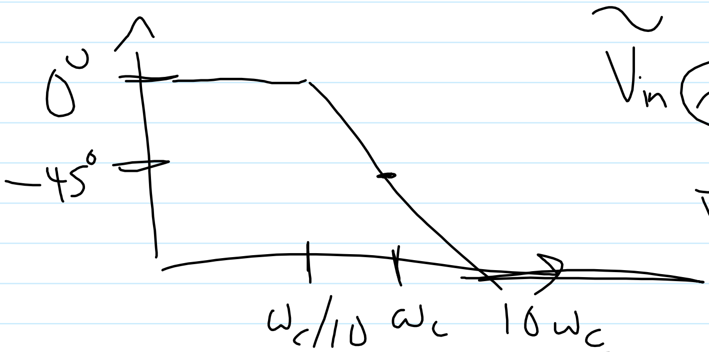
$$\angle \frac{Z_1}{Z_2} = \angle \frac{|Z_1| e^{j\theta_1}}{|Z_2| e^{j\theta_2}} = \angle e^{j(\theta_1-\theta_2)}$$

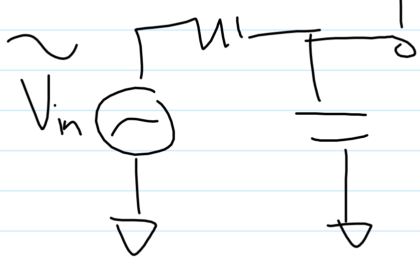
$$= \theta_1 - \theta_2 = \angle Z_1 - \angle Z_2$$

$$\angle Z_1 Z_2 = \angle e^{j\theta_1} e^{j\theta_2} = \angle e^{j(\theta_1+\theta_2)}$$

$$= \theta_1 + \theta_2 = \angle Z_1 + \angle Z_2$$

Play the same trick: add and subtract constituent plots!





III. Plotting Bode Plots

Given factored form of transfer fn:

1. Draw out Bode plots for each component function $\Longrightarrow$ table
2. Add the plots together!
   — b/c log, exponent properties

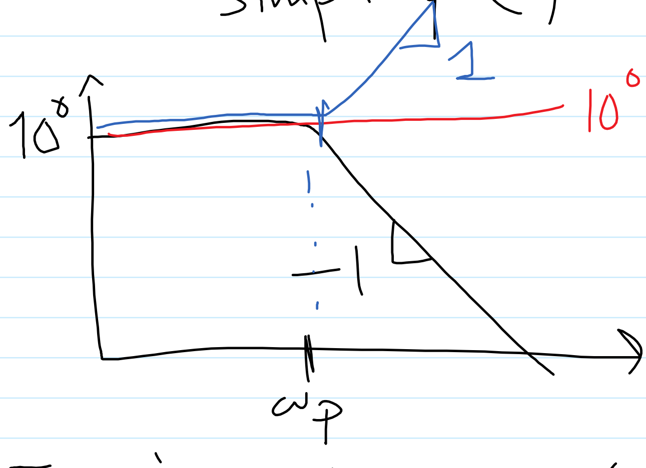
   magnitude — phase

EX1: $1 + \frac{j\omega}{\omega_p} = H'(j\omega)$

("simple zero")

$$H'(j\omega) = \frac{1}{\frac{1}{1+\frac{j\omega}{\omega_p}}} = \frac{1}{H(j\omega)}$$

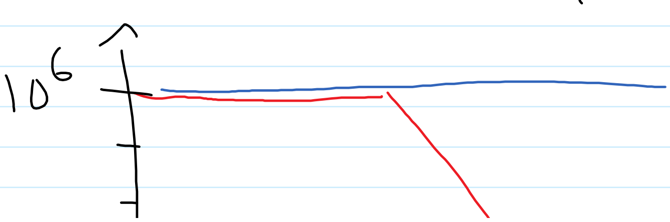
Simple pole, aka LPF

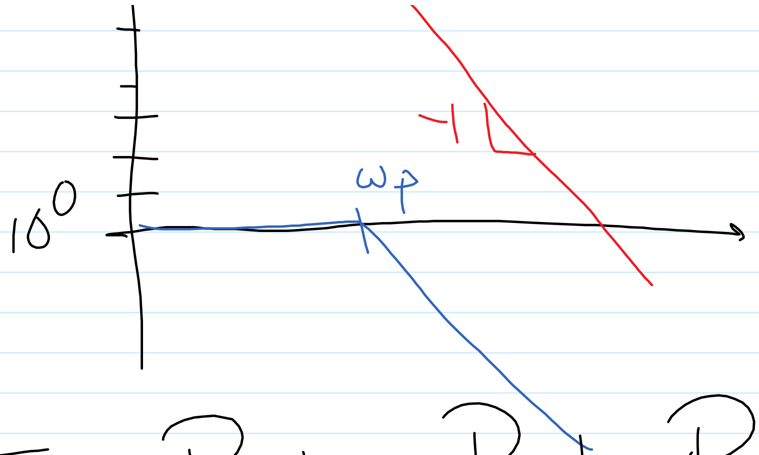


$$\frac{1}{\left(\frac{1}{1+j\frac{\omega}{\omega_p}}\right)}$$

$\log(A/B) = \log A - \log B$

EX2: $H(j\omega) = 10^6 \frac{1}{1 + j\frac{\omega}{\omega_p}}$





IV. Reading Bode Plots

- yesterday, we built intuition that pole ω_p is where slope goes down
- zero does the opposite

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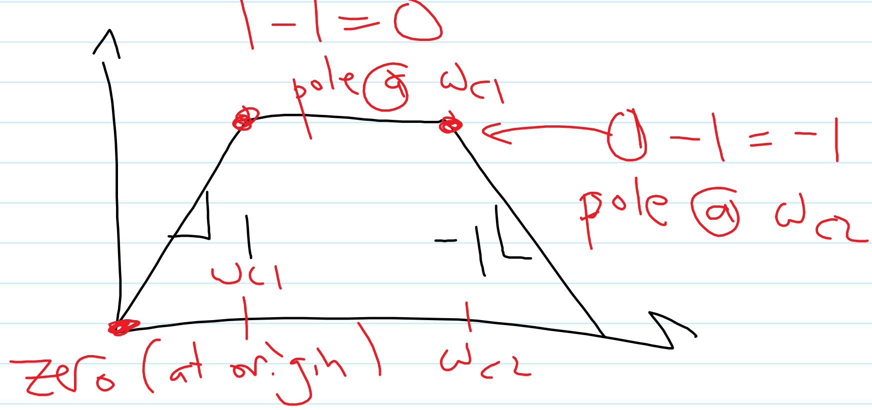
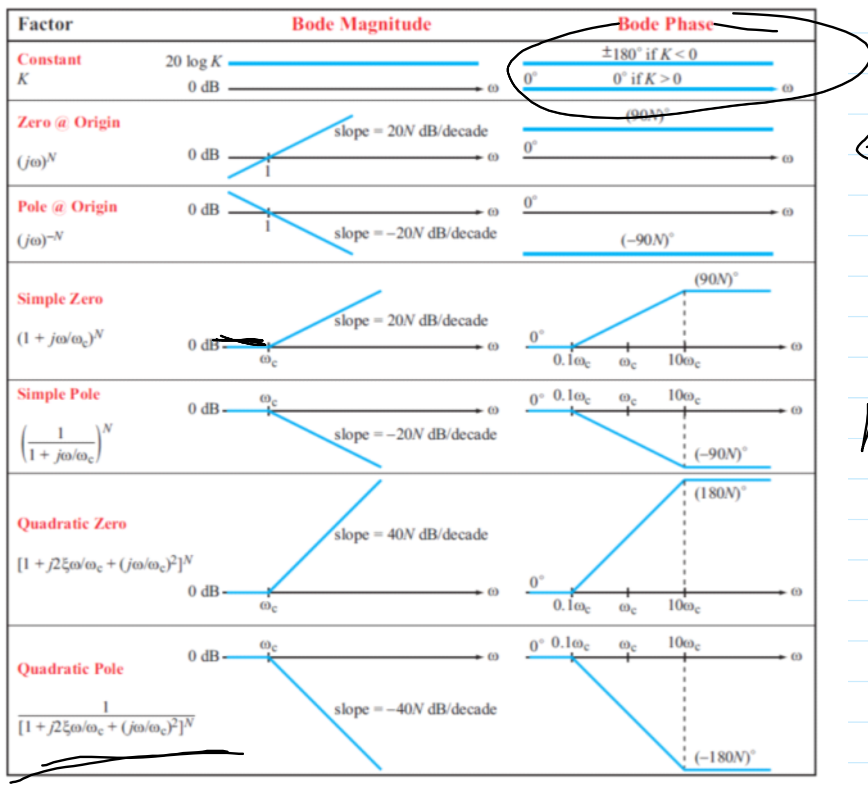
slope goes down by -1 $\Longrightarrow$ pole at that value ω_p

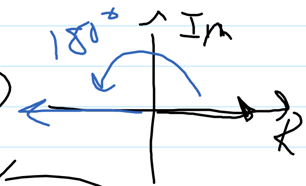
slope goes up by $+1$ $\Longrightarrow$ zero at that value ω_z



Dis 2D Worksheet

Wednesday, July 1, 2020 4:36 PM





20N dB/decade

↓

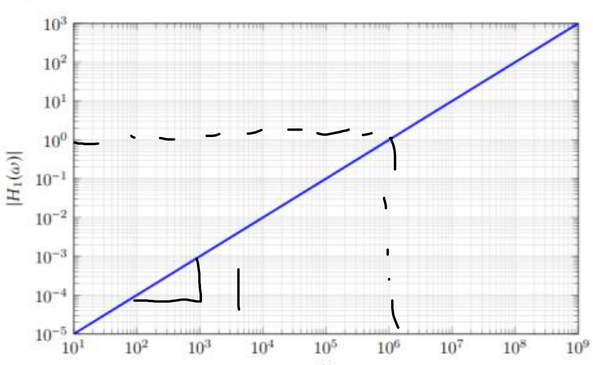
N orders of magnitude per decade

↓

slope of N

1 Bode Plot Practice

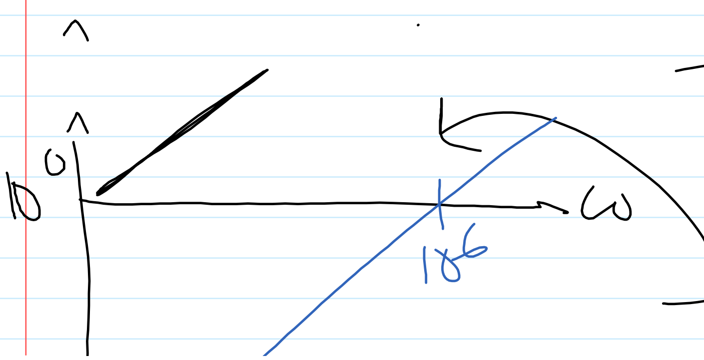
a) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_1(\omega)$ would result in this plot?**

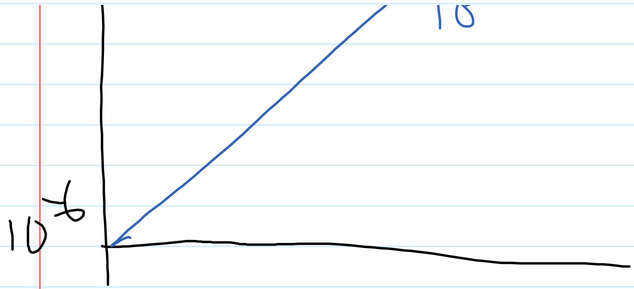


- slope is going up
  $\Longrightarrow$ zero

$|j\omega| = \omega \rightarrow$ goes up w/ ω (pos. slope)

- Slope is 1
  $\Longrightarrow$ single zero
- Doesn't look like it ever goes flat
  $\Longrightarrow$ zero at origin
- $|H_1(j10^6)| = 1$

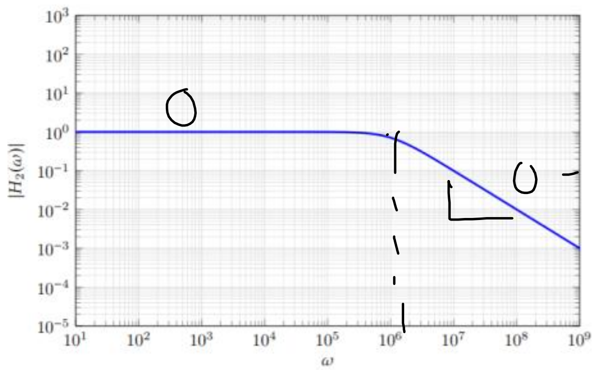




$- \ |H_1(j10^6)| = 1$

$\Longrightarrow \boxed{H_1(j10^6) = \dfrac{j\omega}{10^6}}$

b) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_2(\omega)$ would result in this plot?**



$0 - 1 = -1$

~ initially $|H| = 1$, scaling is 1

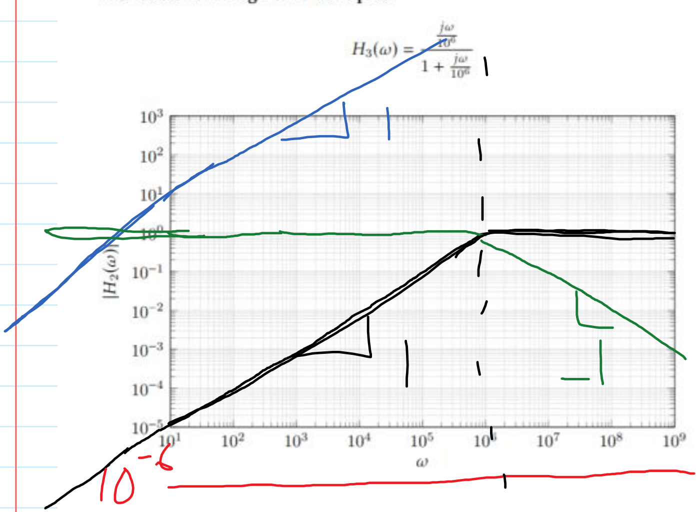
– goes down by 1

$\longrightarrow$ pole at 10^6

$$\boxed{H_2(\omega) = \frac{1}{1 + j\frac{\omega}{10^6}}}$$

c) **Identify the locations of the poles and zeroes in the following transfer function. Then sketch the magnitude Bode plot.**

$$H_3(\omega) = \frac{\frac{j\omega}{10^6}}{1 + \frac{j\omega}{10^6}}$$



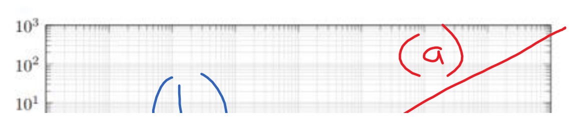
Method: table!

$$H_3(\omega) = \frac{1}{10^6} \times (j\omega) \times \frac{1}{1 + j\frac{\omega}{10^6}}$$

High Pass filter!

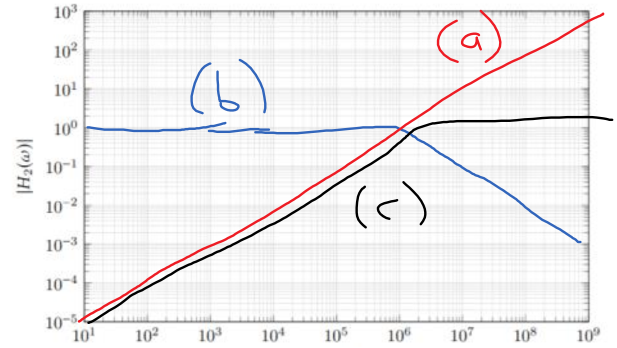
c) **Identify the locations of the poles and zeroes in the following transfer function. Then sketch the magnitude Bode plot.**

$$H_3(\omega) = \frac{\frac{j\omega}{10^6}}{1 + \frac{j\omega}{10^6}}$$



Alternative!

Notice (c) = (a) × (b)

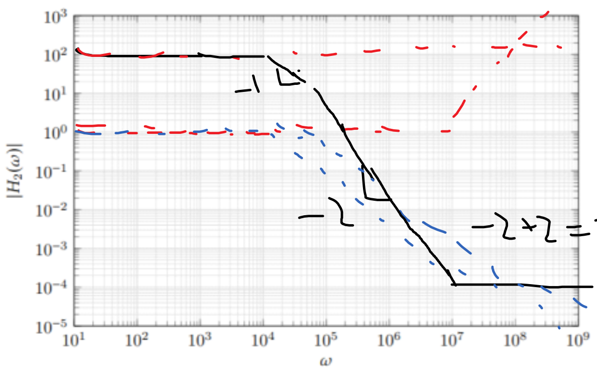


Notice (c) = (a) × (b)

d) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. Then sketch the magnitude Bode plot.**

$$H_4(\omega) = 100\frac{(1+\frac{j\omega}{10^7})^2}{(1+\frac{j\omega}{10^4})(1+\frac{j\omega}{5\times10^4})}$$

Find ω_p's, ω_z's:



$$H_4(\omega) = 100 \times \left(1+\frac{j\omega}{10^7}\right)\left(1+\frac{j\omega}{10^7}\right) \times \frac{1}{\left(1+\frac{j\omega}{10^4}\right)} \times \frac{1}{\left(1+\frac{j\omega}{5\times10^4}\right)}$$

zeroes: 2 at $\omega_z = 10^7$ rad/s

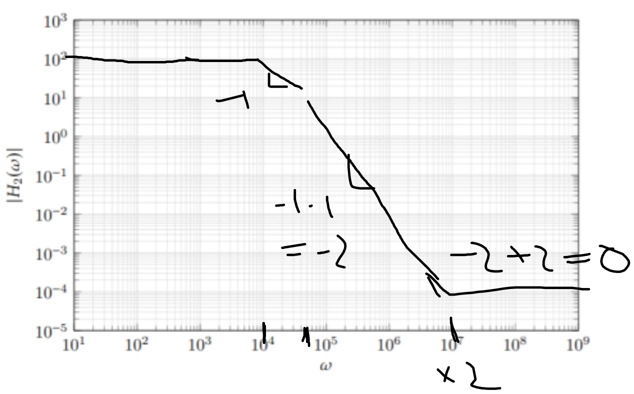
poles: $\omega_{p1} = 10^4$ rad/s

$\omega_{p2} = 5\times10^4$ rad/s

$K = 100 = 10^2$

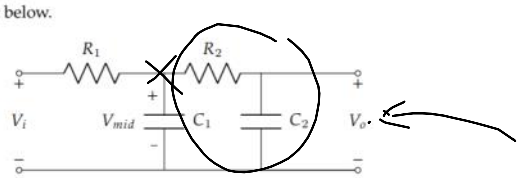
d) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. Then sketch the magnitude Bode plot.**

$$H_4(\omega) = 100\frac{(1+\frac{j\omega}{10^7})^2}{(1+\frac{j\omega}{10^4})(1+\frac{j\omega}{5\times10^4})}$$



2 Transfer functions and why loading is annoying

Consider the circuit below.



$$H(j\omega) = \frac{\tilde{V}_o}{\tilde{V}_i}$$

The circuit has an input phasor voltage $\widetilde{V}_i$ at frequency ω rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage $\widetilde{V}_o$ at output terminals.

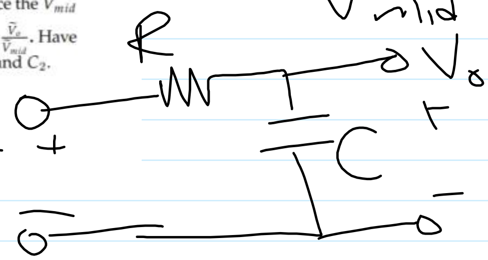
a) We are going to construct the transfer function $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$ in two steps. We will

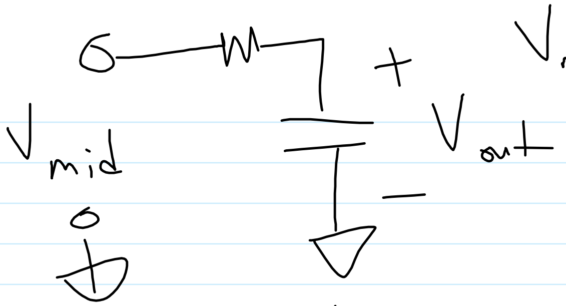
$H_-(j\omega) = \tilde{V}_o$

terminals shown in the illustration above, causing an output phasor voltage $\widetilde{V}_o$ at output terminals.

a) We are going to construct the transfer function $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$ in two steps. We will compute two intermediate transfer functions, $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$ and $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$. Then, we will find the overall transfer function as the product of these two intermediate transfer functions, i.e. $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$. This approach is valid since the $\widetilde{V}_{mid}$ cancel. For the first step, **find the intermediate transfer function** $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$. Have your expression be in terms of Z_{R2} and Z_{C2}, that is the impedances of R_2 and C_2.

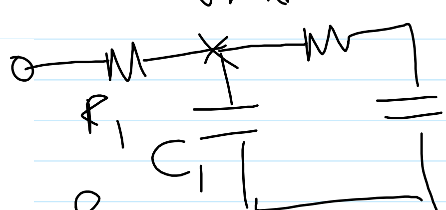
$$H_2(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$$





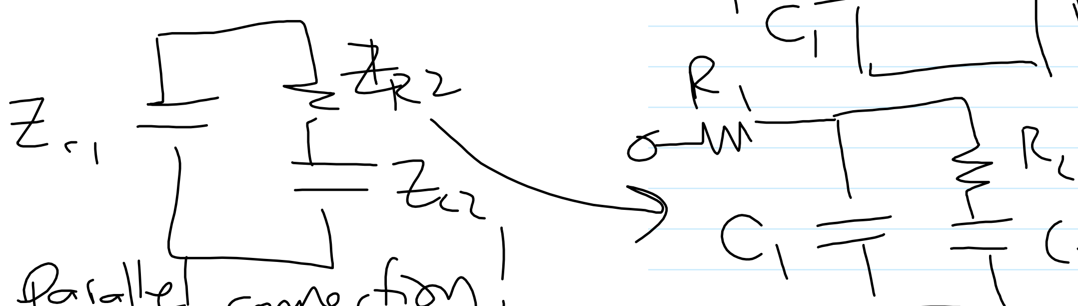
$$\frac{\widetilde{V}_{out}}{\widetilde{V}_{mid}} = \frac{Z_{C2}}{Z_{C2} + Z_{R2}} = \frac{\frac{1}{j\omega C_2}}{R + \frac{1}{j\omega C_2}} = \underline{\underline{\frac{1}{1 + j\omega R_2 C_2}}}$$

b) Now, **compute the other intermediate transfer function** $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$. Have your expression be in terms of Z_{R1}, Z_{R2}, Z_{C1}, and Z_{C2}. (i.e. Don't forget to consider the impact of loading by R_2 and C_2 in this transfer function.)



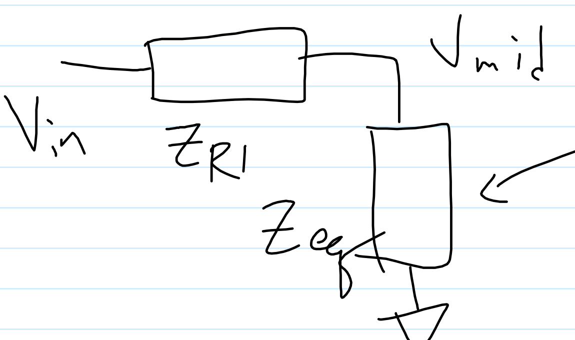
$R_1 + R_2$

$R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$



parallel connection!

$\Longrightarrow$ collapse into equivalent impedance



$$Z_{eq} = Z_{C1} \| (Z_{C2} + Z_{R2})$$

$$\frac{\widetilde{V}_{mid}}{\widetilde{V}} = \frac{Z_{eq}}{} = \frac{\left(\frac{1}{j\omega C_1}\right) \| \left(\frac{1}{j\omega C_2} + R_2\right)}{}$$

$$\frac{V_{mid}}{V_{in}} = \frac{Z_{eq}}{Z_{eq} + Z_{R1}} = \frac{\left(\frac{1}{j\omega C_1}\right) \parallel \left(\frac{1}{j\omega C_2} + R_2\right)}{\left(\frac{1}{j\omega C_1} \parallel \left(\frac{1}{j\omega C_2} + R_2\right)\right) + R_1}$$

$$= \frac{\dfrac{\frac{1}{j\omega C_1} \times \left(\frac{1}{j\omega C_2} + R_2\right)}{\frac{1}{j\omega C_1} + \cancel{\frac{1}{j\omega C_1} + R_2}}}{\dfrac{\frac{1}{j\omega C_1} \times \left(\frac{1}{j\omega C_2} + R_2\right)}{\frac{1}{j\omega C_1} + \cancel{\frac{1}{j\omega C_2} + R_2}} + R_1} \times \frac{\left(\frac{1}{j\omega C_1} + \cancel{\frac{1}{j\omega C_2} + R_2}\right)}{\left(\frac{1}{j\omega C_2} + \cancel{\frac{1}{j\omega C_2} + R_2}\right)}$$

$$= \frac{\left(\frac{1}{j\omega C_2} + R_2\right)\frac{1}{j\omega C_1}}{\left(\frac{1}{j\omega C_2} + R_2\right)\frac{1}{j\omega C_1} + R_1 R_2 + \frac{R_1}{j\omega C_1} + \frac{R_2}{j\omega C_2}} \times \frac{(j\omega C_1)(j\omega C_2)}{(j\omega C_1)(j\omega C_2)}$$

$$= \frac{j\omega C_2\left(\frac{1}{j\omega C_2} + R_2\right)}{\left(\frac{1}{j\omega C_2} + R_2\right) j\omega C_2 + (j\omega)^2 R_1 R_2 C_1 C_2 + j\omega C_2 R_1 + j\omega C_1 R_1}$$

$$\Rightarrow H_1(\omega) = \frac{1 + j\omega R_2 C_2}{1 + j\omega(R_1 C_1 + R_2 C_2 + R_1 C_2) + (j\omega)^2 R_1 R_2 C_1 C_2}$$

c) Then, **use these two intermediate transfer functions to calculate the overall transfer function** as $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$.

$$H(j\omega) = \frac{1}{1 + j\omega(R_1C_1 + R_2C_2 + R_1C_2) + (j\omega)^2 R_1R_2C_1C_2}$$

d) Sometimes it is useful to collect all the frequency dependence into one place and to figure out how to think about what scale might be somewhat natural for the frequency. **Obtain an expression for** $H(\omega) = \widetilde{V}_o/\widetilde{V}_i$ **in the form**

$$H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = \frac{1}{1 + 2\xi\frac{j\omega}{\omega_c} + \frac{(j\omega)^2}{\omega_c^2}},$$

given that $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, $C_1 = \frac{9}{2}\,\text{F}$, and $C_2 = 1\,\text{F}$. What are the values of ξ and ω_c?

$$R_1C_1 = 2\Omega \times \frac{9}{2}F = 9s$$

$$R_2C_2 = 4\Omega \times 1F = 4s$$

$$R_1C_2 = 2\Omega \times 1F = 2s$$

$$\Rightarrow H(j\omega) = \frac{1}{1 + 15j\omega + (j\omega)^2 36} \leftarrow \omega_c^2 = \frac{1}{36}$$

$$\omega_c = \frac{1}{6}$$

$$\boxed{\xi = \frac{5}{4},\ \omega_c = \frac{1}{6}\ \text{rad/s}}$$

e) **For the previous case, what is the magnitude of the transfer function at the** $\omega = \omega_c$ **you calculated?** ← corner freq

This is here so that you can see that just because we called it ω_c doesn't mean that the amplitude here is $\frac{1}{\sqrt{2}}$.

$$H(\omega) = \frac{1}{\cancel{1} + 2\xi\frac{j\omega_c}{\omega_c} + \cancel{\frac{(j\omega_c)^2}{\omega_c^2}}} \leftarrow -1$$

$$= \frac{1}{2\xi j}$$

$$|H(\omega_c)| = \frac{1}{2\xi} = \frac{1}{2} \times \frac{1}{5} = \frac{2}{5} = \underline{\underline{0.4}}$$

quadratic $\Longrightarrow$ we would expect

$$\frac{1}{\left(1+j\frac{\omega}{\omega_c}\right)^2} \xrightarrow{\omega=\omega_c} \frac{1}{(1+j)^2}$$

$$\left|H_{\substack{perf\\quadratic}}(\omega_c)\right| = \frac{1}{|1+2j-1|} = \frac{1}{2}$$

B/c of loading, $|H(\omega_c)| \neq |H_{\substack{perf\\quad}}(\omega_c)|$

f) We can express the transfer function $H(\omega)$ in the polar form. That is,

$$H(\omega) = M(\omega)e^{j\phi(\omega)}$$

The functions $M(\omega)$ and $\phi(\omega)$ are the magnitude and the phase angle of $H(\omega)$, respectively. **Write down $M(\omega)$ and $\phi(\omega)$ using the transfer function you derived in part (b).**

$$H(j\omega) = \frac{1}{1+15j\omega-36\omega^2} \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$M(\omega) = |H(j\omega)| = \frac{1}{\sqrt{(1-36\omega^2)^2+(15\omega)^2}}$$

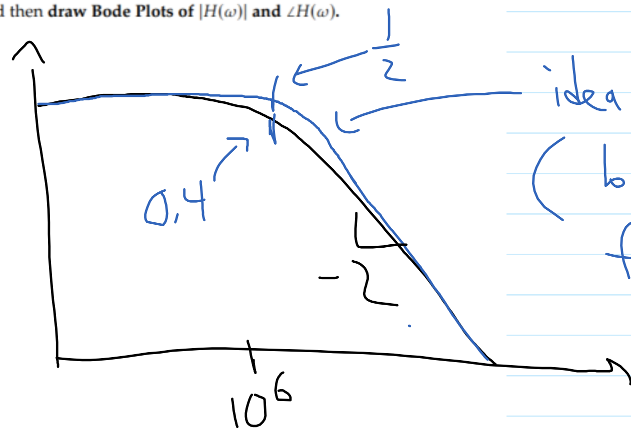
$$1+15j\omega-36\omega^2 = (1-36\omega^2)+j(15\omega) = x+jy$$

$$\measuredangle H(j\omega) = \measuredangle 1 - \measuredangle\left((1-36\omega^2)+j(15\omega)\right)$$

$$= 0 - \measuredangle\left((1-36\omega^2)+j(15\omega)\right)$$

$$= -\operatorname{atan2}(15\omega, 1-36\omega^2)$$

g) Use a computer and then **draw Bode Plots of** $|H(\omega)|$ **and** $\angle H(\omega)$.



ideal cascade
(buffer in b/w
filters)

$\xi = 0, \quad |H(\omega_c)| = \frac{1}{2\xi 0} = \frac{1}{0} \longrightarrow \infty$