Dis 3A Notes

Sunday, July 5, 2020 8:25 PM

Multivariate Diff Eq

* Overview (Big Picture)
* Multivariate Diff Eq
  - Solving using diagonalization
  - Worksheet

I. Logistics

- HW2 due Tues 7/7

HW3 assigned soon

- HW Party: temp. moved to M, Tue
  $\Longrightarrow$ F 1–5 pm also

II. Overview

$\Longrightarrow$ What's in your toolbox right now?

(1) 1st order diff eq (various cases)

$$\frac{dx(t)}{dt} + ax(t) = b(t) \longleftarrow \text{input}$$

a) Homogeneous solution ($b(t) = 0$)

$$\Rightarrow \frac{dx(t)}{dt} + ax(t) = 0$$

$$x(t) = X_0 e^{-at}$$

i) guess-and-check

ii) Calculus

b) constant input: $b(t) = b$

$$\frac{dx(t)}{dt} + ax(t) = b$$

$$\Longrightarrow x(t) = \underbrace{\frac{b}{a}}_{x_p} + \underbrace{Ce^{-at}}_{x_h}$$

i) guess-and-check

ii) change of variables

$$\tilde{x} = x - \frac{b}{a}$$

(math trick)

iii) Calculus

c) time-dependent input $b(t)$

$$\frac{dx(t)}{dt} + ax(t) = b(t)$$

$$\frac{dx(t)}{dt} + a x(t) = b(t)$$

$$\Rightarrow \quad x(t) = \underbrace{e^{-a(t-t_0)} x_0}_{x_h} + \underbrace{\int_{t_0}^{t} e^{-a(t-t')} b(t')\,dt'}_{x_p}$$

disgusting

for $x(t_0) = x_0$

i) math tricks

ii) guess-and-check

$\Longrightarrow$ special class of time-varying inputs: sinusoids

2 Phasors

$$v(t) = V_0 \cos(\omega t + \phi) \Longrightarrow \tilde{V}_0 = V_0 e^{j\phi}$$

corresponds to

$$v(t) = V_0 \cos(\omega t + \phi) = \mathrm{Re}\left[\tilde{V}_0 e^{j\omega t}\right]$$

equality

$$= \frac{1}{2}\left[\tilde{V}_0 e^{j\omega t} + \overline{\tilde{V}_0 e^{j\omega t}}\right]$$

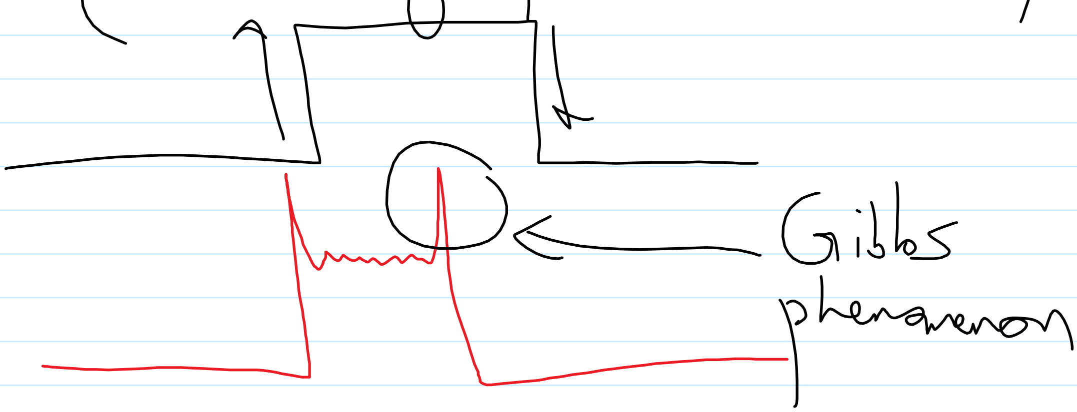
⟹ lets us find the <u>steady-state response to a sinusoidal input</u>

(w/o ever writing a diff eq!)

- ✱ transfer functions
- ✱ filters
- ✱ Bode plots

But phasor analysis is limited to sinusoidal inputs! (or lin. comb. of sinusoids)

⟹ e.g. step functions, square waves (fns w/ jump discontinuities)



③ Multivariate Diff Eqs

$\Longrightarrow$ back to time domain...

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

differentiation (linear operator) represented as matrix!

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

coupled diff eqs, annoying to solve

Would prefer:

$$\frac{d}{dt}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

$\Longrightarrow$ 2 decoupled diff eqs

$$\tilde{x}_1(t) = \tilde{x}_1(0)\, e^{A_{11} t}$$

$$\tilde{x}_2(t) = \tilde{x}_2(0)\, e^{A_{22} t}$$

$\Longrightarrow$ How do we get this form?

A: diagonalization

$$A = V\Lambda V^{-1} \implies \Lambda = V^{-1}AV$$

matrix w/ eigenvecs as columns

diagonal matrix of eigenvalues

$\implies$ onto the worksheet!

Dis 3A Worksheet

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1 Diagonalization

Consider an $n \times n$ matrix A that has n linearly independent eigenvalue/eigenvector pairs $(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)$ that can be put into a matrices V and Λ.

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

a) Show that $AV = V\Lambda$.

$$A\vec{v}_1 = \lambda_1\vec{v}_1$$
$$A\vec{v}_2 = \lambda_2\vec{v}_2$$
$$\vdots$$
$$A\vec{v}_n = \lambda_n\vec{v}_n$$

$$A\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \ldots & A\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= AV$$

take a closer look

$$\begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \ldots & A\vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \ldots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$\lambda_1\vec{v}_1 = \lambda_1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \ldots + 0\vec{v}_n$$

$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = V\vec{\lambda}_1$$

$$AV = \begin{bmatrix} | & | & & | \\ V\vec{\lambda}_1 & V\vec{\lambda}_2 & \ldots & V\vec{\lambda}_n \\ | & | & & | \end{bmatrix} = V\begin{bmatrix} | & | & & | \\ \vec{\lambda}_1 & \vec{\lambda}_2 & \ldots & \vec{\lambda}_n \\ | & | & & | \end{bmatrix}$$

Λ

$$AV = V\Lambda$$

b) Use the fact in part (a) to conclude that $A = V\Lambda V^{-1}$.

$$AV = V\Lambda$$

$AV = V\Lambda$

V is invertible

$\implies A = V \Lambda V^{-1}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \leftarrow$ can't be diagonalized!

"defective matrix"

$\lambda = 1, 1 \leftarrow \vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2 Systems of Differential Equations

Consider a system of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -4x_1(t) + x_2(t) \quad (1)$$

$$\frac{d}{dt}x_2(t) = 2x_1(t) - 3x_2(t) \quad (2)$$

with initial conditions $x_1(0) = 3$ and $x_2(0) = 3$.

a) Write out the differential equations and initial conditions in matrix/vector form.

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

b) Find the eigenvalues $\lambda_1,\ \lambda_2$ and eigenspaces for the differential equation matrix above.

$\lambda_1 = -5,\ \lambda_2 = -2$

$$\det(A - \lambda I) = \det\begin{pmatrix} -4-\lambda & 1 \\ 2 & -3-\lambda \end{pmatrix} = 0$$

$$\implies (\lambda+4)(\lambda+3) - 2 = \lambda^2 + 7\lambda + 12 - 2$$

$$= \lambda^2 + 7\lambda + 10 = 0$$

$\lambda = -5$

$$= \lambda^2 + 7\lambda + 10 = 0$$

$$\Longrightarrow (\lambda+5)(\lambda+2) = 0 \longrightarrow \lambda_1 = -5, \quad \lambda_2 = -2$$

$\lambda = -5$:

$$\begin{pmatrix} -4+5 & 1 \\ 2 & -3+5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

By inspection, $\vec{v_1} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 + x_2 = 0$

Pick $x_1 = \alpha$ (free variable)

$$x_2 = -x_1 = -\alpha$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = -2$: $\begin{pmatrix} -4+2 & 1 \\ 2 & -3+2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$

$$\vec{v_2} = \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{\begin{array}{l} \lambda_1 = -5, \ \vec{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda_2 = -2, \ \vec{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}}$$

$$\boxed{\begin{aligned}\lambda_1 &= -5, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \lambda_2 &= -2, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\end{aligned}}$$

c) Use the diagonalization of $A = V\Lambda V^{-1}$ to express the differential equation in terms of a new variables $z_1(t)$, $z_2(t)$. Remember to find the new initial conditions $z_1(0)$ and $z_2(0)$. (These variables represent eigenbasis-aligned coordinates.)

$$\frac{d}{dt}\vec{x} = A\vec{x} = V\Lambda\left(V^{-1}\vec{x}\right)$$

$$\frac{d}{dt}\vec{z} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\vec{z}$$

$$\vec{z}(t) = V^{-1}\vec{x}(t)$$

$$V^{-1}\left(\frac{d\vec{x}}{dt} = V\Lambda\vec{z}(t)\right) \Longrightarrow V^{-1}\frac{d\vec{x}}{dt} = \Lambda\vec{z}(t)$$

V^{-1} is time independent, and $\frac{d}{dt}$ is linear

$$\Longrightarrow \frac{d}{dt}\big(\underbrace{V^{-1}\vec{x}}_{\vec{z}(t)}\big) = \underline{\Lambda}\,\vec{z}(t)$$

$$\boxed{\frac{d}{dt}\vec{z}(t) = \underline{\Lambda}\,\vec{z}(t)}$$

$$\begin{bmatrix} -5 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\frac{dz_1}{dt} = -5z_1$$

$$\frac{dz_2}{dt} = -2z_2$$

d) Solve the differential equation for $z_i(t)$ in the eigenbasis.

$$\vec{z}(t) = \begin{bmatrix} z_1(0)e^{\lambda_1 t} \\ z_2(0)e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} z_1(0)e^{-5t} \\ z_2(0)e^{-2t} \end{bmatrix}$$

$$\vec{z}(t) = V^{-1}\vec{x}(t)$$

$$\Rightarrow \vec{z}(0) = V^{-1}\vec{x}(0) = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \leftarrow \vec{x}(0)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \quad \vec{z}(0) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{\begin{aligned} z_1(t) &= 1e^{-5t} \\ z_2(t) &= 2e^{-2t} \end{aligned}}$$

e) Convert your solution back into the original coordinates to find $x_i(t)$.

$$\vec{x}(t) = C\,[\underbrace{V^{-1}\vec{x}(t)}_{\vec{z}(t)}]$$

$$= V[V^{-1}\vec{x}(t)] = \vec{x}(t)$$

$$\vec{x}(t) = V\vec{z}(t) = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} e^{-5t} \\ 2e^{-2t} \end{pmatrix}$$

$$\boxed{\vec{x}(t) = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-5t} + 4e^{-2t} \end{bmatrix}}$$

$$\boxed{\vec{x}(t) = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-5t} + 4e^{-2t} \end{bmatrix}}$$

f) We can solve this equation using a slightly shorter approach by observing that the solutions for $x_i(t)$ will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

lin comb. of exponentials

where λ_k is an eigenvalue of our differential equation relation matrix A.

Since we have observed that the solutions will include $e^{\lambda_i t}$ terms, once we have found the eigenvalues for our differential equation matrix, we can guess the forms of the $x_i(t)$ as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

guess

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are all constants.

Take the derivative to write out

$$\begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix}.$$

and connect this to the given differential equation.

Solve for $x_i(t)$ from this form of the derivative.

4 unknowns $\longrightarrow$ 4 lin. ind. eqns needed

Use initial conditions

$$\vec{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix} \Bigg|_{t=0}$$

$$\vec{x}(0) = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

2 eqns

$$\frac{d\vec{x}}{dt} = A\vec{x} = \frac{d}{dt}\begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$$

Evaluate at $t=0$

$$\Rightarrow \left.\begin{bmatrix} -4x_1(t) + x_2(t) \\ 2x_1(t) - 3x_2(t) \end{bmatrix}\right|_{t=0} = \left.\begin{bmatrix} \alpha_1\lambda_1 e^{\lambda_1 t} + \alpha_2\lambda_2 e^{\lambda_2 t} \\ \beta_1\lambda_1 e^{\lambda_1 t} + \beta_2\lambda_2 e^{\lambda_2 t} \end{bmatrix}\right|_{t=0}$$

$$\begin{bmatrix} -9 \\ -3 \end{bmatrix} = \begin{bmatrix} \alpha_1\lambda_1 + \alpha_2\lambda_2 \\ \beta_1\lambda_1 + \beta_2\lambda_2 \end{bmatrix} \leftarrow \text{other 2 eqns}$$

Plug in $\lambda_1 = -5$, $\lambda_2 = -2$

$$\left.\begin{aligned} \alpha_1 + \alpha_2 &= 3 \\ \beta_1 + \beta_2 &= 3 \\ -5\alpha_1 - 2\alpha_2 &= -9 \\ -5\beta_1 - 2\beta_2 &= -3 \end{aligned}\right] \rightarrow \begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= 2 \\ \beta_1 &= -1 \\ \beta_2 &= 4 \end{aligned}$$

$$\Rightarrow \boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-5t} + 2e^{-2t} \\ -e^{-5t} + 4e^{-2t} \end{bmatrix}}$$

$\Rightarrow$ How do you solve a second order diff eq?

$$\to a_0 x + a_1 \frac{dx}{dt} + a_2 \frac{d^2x}{dt^2} = 0$$

Pick $x, \frac{dx}{dt}$ as variables

"x_1" "x_2"

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \end{bmatrix} = \begin{bmatrix} \frac{dx}{dt} \\ -\frac{a_0}{a_2}x - \frac{a_1}{a_2}\frac{dx}{dt} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{a_0}{a_2} & -\frac{a_1}{a_2} \end{bmatrix} \begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix}$$