Dis 3B Notes

Tuesday, July 7, 2020 12:14 AM

Multivariate Diff Eq II

* Change of Basis
* Diagonalization & Change of Basis
* Circuit Examples of Multivariate Diff Eq
  - LC Tank
  - Driven RLC Circuit

(Worksheet)

I. Change of Basis

Consider $\vec{x}_e = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

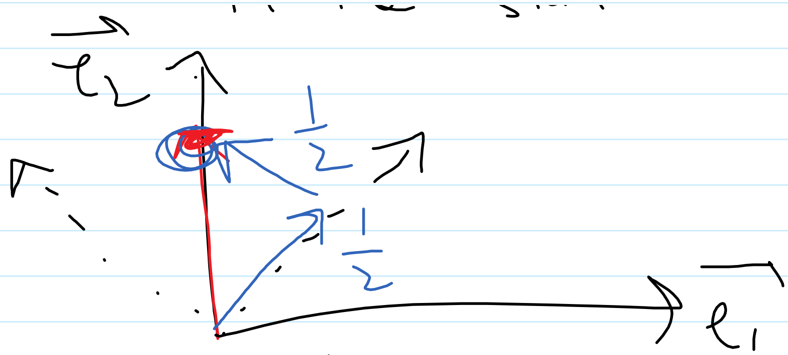
$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(standard basis)

$$\vec{x}_e = 0\vec{e}_1 + 1\vec{e}_2$$

$\vec{x}_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the coordinates of $\vec{x}$ in the standard basis

$\vec{e}_1, \ldots$



Consider $\vec{b_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{b_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\vec{x_e} = \frac{1}{2}\vec{b_1} + \frac{1}{2}\vec{b_2}$$

$\vec{x_b} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ is coordinates of $\vec{x}$ in the B basis

Same point, different representation!

How do we relate those 2 representations

$\Longrightarrow$ how to do a "change of basis"

$$\vec{x_e} = \frac{1}{2}\vec{b_1} + \frac{1}{2}\vec{b_2} = \begin{pmatrix} | & | \\ \vec{b_1} & \vec{b_2} \\ | & | \end{pmatrix} \underbrace{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}}_{\vec{x_b}}$$

$$\vec{x_e} = B\,\vec{x_b}$$

$\vec{x_b} = B^{-1}\vec{x_e}$

$$\boxed{\vec{x}_b = B^{-1}\vec{x}_e}$$

Change of basis, std $\to$ B basis

How might we change from any basis A (not necessarily std basis) to basis B?

$$\begin{tikzcd}
\vec{x}_a \arrow[rr, "B^{-1}A"] \arrow[dr, "A"'] & & \vec{x}_b \\
& \vec{x}_e \arrow[ur, "B^{-1}"'] &
\end{tikzcd}$$

$$\boxed{\begin{aligned} \vec{x}_b &= B^{-1}A\vec{x}_a \\ C_{A\to B} &= B^{-1}A \end{aligned}}$$

In fact, earlier case is special case.

$$A = I$$

$$C_{A \to B} = B^{-1}A = B^{-1}I = B^{-1}$$

As before

II. Diagonalization & Change of Basis

$$\vec{y} = A\vec{x} = \underline{V \Lambda V^{+}} \vec{x}$$

$$\begin{tikzcd}
\vec{x} \arrow[rr, "A"] \arrow[d, "V^{-1} \; (1)"'] & & \vec{y} = A\vec{x} \\
\vec{x}_v \arrow[rr, "(2) \; \Lambda"] & & \vec{y}_v \arrow[u, "(3) \; V"']
\end{tikzcd}$$

(1) Change to eigenbasis — aligned coordinates $\vec{x}_v$

(2) Scale w/ diagonal matrix Λ

(3) Transform result back to std basis

: $\vec{y}_v \longrightarrow \vec{y}$

; $\vec{y}_v \longrightarrow \vec{y}$

Step 2 in more detail!

$$A\vec{x} = \vec{y}$$
$$\Lambda \vec{x}_v = \vec{y}_v$$

— very similar looking...

Suggests that Λ is the representation of A in eigenbasis

$\Rightarrow$ How do we get Λ?

$$A = V\Lambda V^{-1} \Longrightarrow \Lambda = V^{-1}AV$$

Look at chart:

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, shift right=2, "V^{-1}"'] & \vec{y} \arrow[d, shift left=2, "\underline{V^{-1}}"] \\
\vec{x}_v \arrow[r, "\Lambda"'] \arrow[u, shift right=2, "\underline{V}"'] & \vec{y}_v \arrow[u, shift left=2, "V"]
\end{tikzcd}$$

$$\Lambda = V^{-1} A V$$

What about other bases?

$$\begin{array}{ccc} \vec{x} & \xrightarrow{\quad A \quad} & \vec{y} = A\vec{x} \\ \Big\downarrow P^{-1} & & \Big\uparrow P \\ \vec{x}_p & \xrightarrow{\quad A_p \quad} & \vec{y}_p \end{array}$$

$$A = P A_p P^{-1} \iff A_p = P^{-1} A P$$

(also known as a similarity transformation)

III. Circuit Examples of Matrix Diff Eq

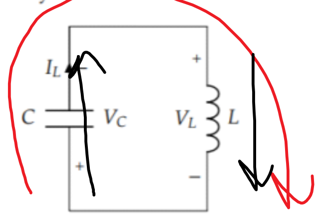
$\longrightarrow$ see worksheet

Dis 3B Worksheet

Tuesday, July 7, 2020 12:14 AM

1 LC Tank

Consider the following circuit like you saw in lecture:



This is sometimes called an LC tank and we will derive its response in this problem. Assume at $t = 0$ we have $V_C(0) = V_S = 1$ V and $\frac{dV_C}{dt}(t = 0) = 0$. Also suppose $L = 9$ nH and $C = 1$ nF.

a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of ~~V_s, L, and~~ C.**

$$\cancel{x_1 = V_C}$$

$$\cancel{x_2 = \frac{dV_C}{dt}}$$

$$x_1(t) = I_L(t)$$

$$x_2(t) = V_C(t)$$

KCL: $I_C(t) - I_L(t) = x_1(t)$

$$x_1(t) = C\frac{dV_C}{dt} = C\frac{d}{dt}x_2(t)$$

KVL: $V_C + V_L = 0$

$$x_2(t) = -V_L(t) = -L\frac{dI_L}{dt} = -L\frac{dx_1}{dt}$$

In sum,

$$x_1(t) = C\frac{dx_2(t)}{dt}$$

$$X_2(t) = -L\frac{dx_1}{dt}(t)$$

b) **Write the system of equations in vector/matrix form with the vector state variable** $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

Find the initial conditions $\vec{x}(0)$.

$$\Rightarrow \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} X_2(t) \\ \frac{1}{C} X_1(t) \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ \vec{x}(0) = \begin{bmatrix} I_L(0) \\ V_C(0) \end{bmatrix} &= \begin{bmatrix} 0 \\ V_S \end{bmatrix} \end{aligned}}$$

c) **Find the eigenvalues of the** A **matrix symbolically.**

$$\det\left[A - \lambda I\right] = \det\begin{bmatrix} -\lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 - \left(-\frac{1}{LC}\right) = \lambda^2 + \frac{1}{LC} = 0$$

$$\boxed{\lambda = \pm \frac{j}{\sqrt{LC}}}$$

d) Recall from yesterday's discussion that solutions for $x_i(t)$ will all be of the form

$$x_i(t) = \sum_k c_k e^{\lambda_k t}$$

where λ_k is an eigenvalue of our differential equation relation matrix A. Thus, we make the following guess for $\vec{x}(t)$:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t} \end{bmatrix}$$

where c_1, c_2, c_3, c_4 are all constants.

Evaluate $\vec{x}(t)$ and $\frac{d\vec{x}}{dt}(t)$ at time $t = 0$ in order to obtain four equations in four unknowns.

4 unknowns $\longrightarrow$ 4 lin. ind. eqns

$$\vec{x}(0) = \begin{bmatrix} 0 \\ V_s \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \end{bmatrix} \Bigg|_{t=0}$$

$$\begin{bmatrix} 0 \\ V_s \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_3 + c_4 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \\ c_3 \lambda_1 e^{\lambda_1 t} + c_4 \lambda_2 e^{\lambda_2 t} \end{bmatrix} \Bigg|_{t=0}$$

$$= A\vec{x} = \begin{bmatrix} -\frac{1}{L} x_2(t) \\ \frac{1}{C} x_1(t) \end{bmatrix} \Bigg|_{t=0}$$

$$\left. \frac{1}{C} x_1(t) \right] \Bigg|_{t=0}$$

$$\begin{bmatrix} c_1\lambda_1 + c_2\lambda_2 \\ c_3\lambda_1 + c_4\lambda_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{L}x_2(0) \\ \frac{1}{C}x_1(0) \end{bmatrix} \begin{matrix} \leftarrow V_S \\ \leftarrow 0 \end{matrix}$$

$$\begin{bmatrix} c_1\lambda_1 + c_2\lambda_2 \\ c_3\lambda_1 + c_4\lambda_2 \end{bmatrix} = \begin{bmatrix} -\frac{V_S}{L} \\ 0 \end{bmatrix}$$

e) **Solve those equations for** c_1, c_2, c_3, c_4 **and plug them into your guess for** $\vec{x}(t)$. What do you notice about the solutions? Are they complex functions? HINT: Remember $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

$$c_1 = -\sqrt{\frac{C}{L}}\frac{V_S}{2j} \qquad c_2 = \sqrt{\frac{C}{L}}\frac{V_S}{2j}$$

$$c_3 = c_4 = \frac{V_S}{2}$$

$$x_1(t) = -\sqrt{\frac{C}{L}}\frac{V_S}{2j}e^{+\frac{j}{\sqrt{LC}}t} + \sqrt{\frac{C}{L}}\frac{V_S}{2j}e^{-\frac{j}{\sqrt{LC}}t}$$

$$= -V_S\sqrt{\frac{C}{L}}\left(\frac{e^{\frac{j}{\sqrt{LC}}t} - e^{\frac{-j}{\sqrt{LC}}t}}{2j}\right)$$

$$\boxed{x_1(t) = I_L(t) = -\sqrt{\frac{C}{L}}V_S \sin\left(\frac{1}{\sqrt{LC}}t\right)}$$

$$\boxed{X_1(t) = I_L(t) = -\sqrt{\frac{C}{L}}\, V_s \sin\left(\frac{1}{\sqrt{LC}}\, t\right)}$$

$$V_2(t) = \frac{V_s}{2}\left[e^{\frac{j}{\sqrt{LC}}t} + e^{\frac{-j}{\sqrt{LC}}t}\right]$$

$$\boxed{V_c(t) = V_s \cos\left(\frac{1}{\sqrt{LC}}\, t\right)}$$

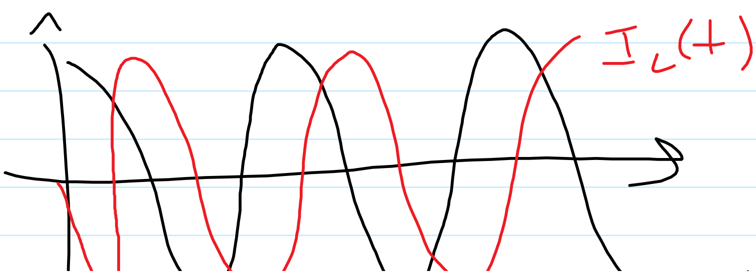
$L = 9\ nH, \quad C = 1\ nF$

$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-9} \times 9 \times 10^{-9}}} = \frac{1}{3\ ns}$$

$$\sqrt{\frac{C}{L}} = \sqrt{\frac{1\ \cancel{nF}}{9\ \cancel{nH}}} = \frac{1}{3}$$

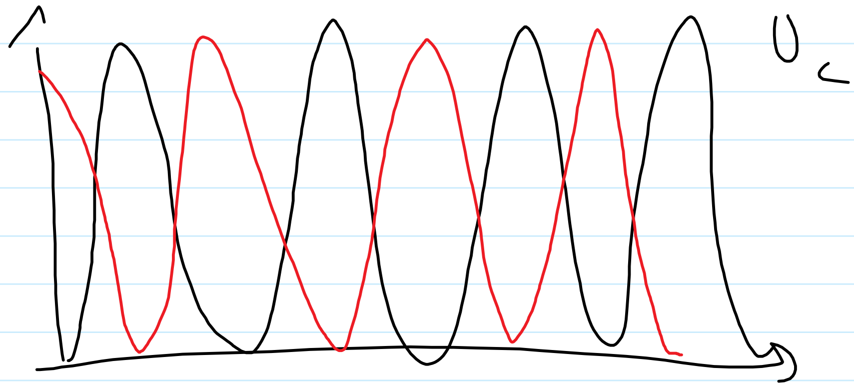
$V_s = 1\ V$

$$\boxed{\begin{aligned} I_L(t) &= -\frac{1}{3} \sin\left(\frac{t}{3\ ns}\right) \quad [A] \\ V_c(t) &= \cos\left(\frac{t}{3\ ns}\right) \quad [V] \end{aligned}}$$

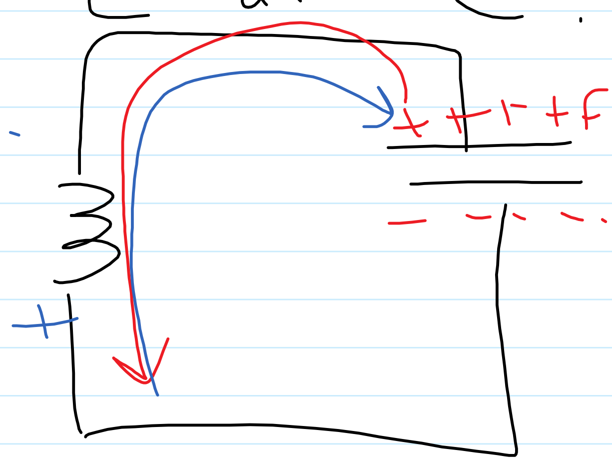


$$U_C = \frac{1}{2} C V_C^2 \propto \cos^2\left(\frac{t}{3\,ns}\right)$$

$$U_L = \frac{1}{2} L I_L^2 \propto \sin^2\left(\frac{t}{3\,ns}\right)$$



Energy oscillating back and forth b/w L and C!



① C dischargs, current flows through L

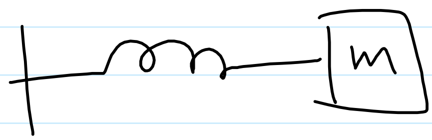
② L builds up voltage b/c of changing current

③ this voltage pushes charge back onto C

Mathematically,

Mathematically, "back onto" C

same as SHO!

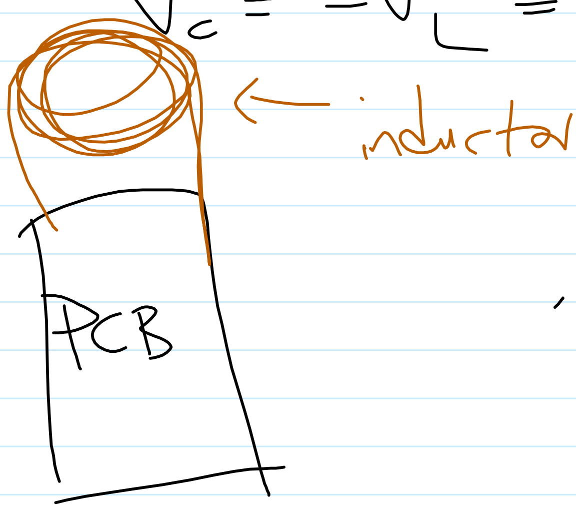


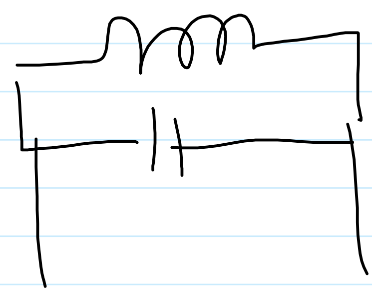
$F = -kx$

$$m\frac{d^2x}{dt^2} = -kx$$

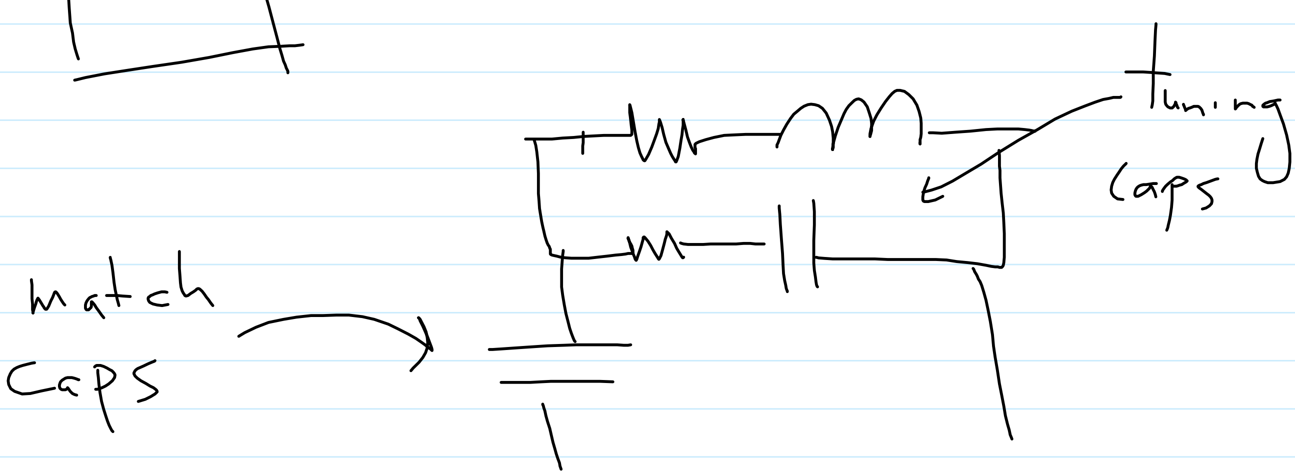
$$I_L(t) = C\frac{dV_c}{dt} = -LC\frac{d^2 I_L}{dt^2}$$

$$V_c = -V_L = -L\frac{dI_L}{dt}$$



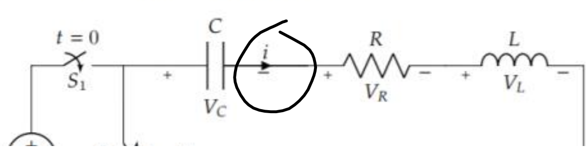


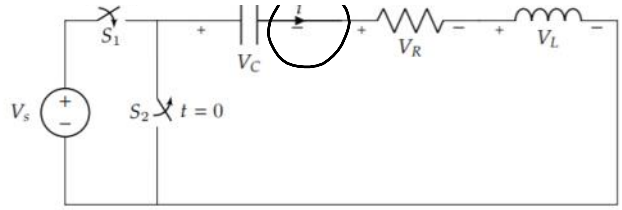
ideal



2 Charging RLC Circuit

Consider the following circuit. Before $t = 0$, switch S_1 is off while S_2 is on. At $t = 0$, both switches flip state (S_1 turns on and S_2 turns off):





$V_s = 0$

$t = 0$

a) Write out the differential equation describing this circuit for $t \geq 0$ in the form:

$$\frac{d^2V_c}{dt^2} + a_1\frac{dV_c}{dt} + a_0V_c = b$$

$$V_s = V_c + V_R + V_L = V_c + iR + L\frac{di}{dt}$$

$$i = C\frac{dV_c}{dt}$$

$$\Rightarrow \boxed{V_s = V_c + RC\frac{dV_c}{dt} + LC\frac{d^2V_c}{dt^2}}$$

$$\boxed{\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c = \frac{V_s}{LC}}$$

b) Find a $\tilde{V}_c$ and substitute it to the previous equation such that

$$\frac{d^2\tilde{V}_c}{dt^2} + a_1\frac{d\tilde{V}_c}{dt} + a_0\tilde{V}_c = 0$$

$$\boxed{\tilde{V}_c = V_c - V_s}$$

$$\rightarrow \frac{d\tilde{V}_c}{dt} = \frac{dV_c}{dt}$$

$$\frac{d\tilde{V}_c}{dt} = \frac{dV_c}{dt} - \cancel{\frac{dV_s}{dt}} \longrightarrow \frac{d^2\tilde{V}_c}{dt^2} = \frac{d^2V_c}{dt^2}$$

$$\frac{d^2\tilde{V}_c}{dt^2} + \frac{R}{L}\frac{d\tilde{V}_c}{dt} + \frac{1}{LC}(V_c) = \frac{1}{LC}(V_s)$$

$$\frac{d^2\tilde{V}_c}{dt^2} + \frac{R}{L}\frac{d\tilde{V}_c}{dt} + \frac{1}{LC}(V_c - V_s) = 0$$

$$\boxed{\frac{d^2\tilde{V}_c}{dt^2} + \frac{R}{L}\frac{d\tilde{V}_c}{dt} + \frac{1}{LC}\tilde{V}_c = 0}$$

$$\Rightarrow \frac{d^2\tilde{V}_c}{dt^2} = -\frac{1}{LC}\tilde{V}_c - \frac{R}{L}\frac{d\tilde{V}_c}{dt}$$

c) Solve for $V_c(t)$ for $t \geq 0$. Use component values $V_s = 4\text{V}$, $C = 2\text{fF}$, $R = 60\text{k}\Omega$, and $L = 1\mu\text{H}$.

$$x_1(t) = \tilde{V}_c$$

$$x_2(t) = \frac{d\tilde{V}_c}{dt}$$

$$\frac{d}{dt}\begin{bmatrix} \tilde{V}_c \\ \frac{d\tilde{V}_c}{dt} \end{bmatrix} = \begin{bmatrix} \frac{d\tilde{V}_c}{dt} \\ -\frac{1}{LC}\tilde{V}_c - \frac{R}{L}\frac{d\tilde{V}_c}{dt} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} \tilde{V}_c \\ \frac{d\tilde{V}_c}{dt} \end{bmatrix}$$

$$\det\begin{bmatrix} -\lambda & 1 \\ -\frac{1}{LC} & -\lambda - \frac{R}{L} \end{bmatrix} = 0$$

$$\Longrightarrow \lambda\left(\lambda + \frac{R}{L}\right) + \frac{1}{LC} = 0$$

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

$$\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$R = 60\,k\Omega \qquad C = 2\,fF$

$L = 1\,mH$

$$\boxed{\begin{aligned} \lambda_1 &= -1 \times 10^{10} \\ \lambda_2 &= -5 \times 10^{10} \end{aligned}}$$

$$\tilde{V}_C(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$\Longrightarrow V_c(t) - V_S = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$V_c(t) = V_s + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Initial Conditions:

steady-state:

$I = 0,\ V = 0$ for $t < 0$

① $I = I_c = C\dfrac{dV_c}{dt}$ ← I_c can't be ∞, so V_c cannot change instantaneously

$$V_c(t<0) = 0 \Longrightarrow V_c(t=0) = 0$$

$$V_c(0) = 0 = V_s + c_1 e^0 + c_2 e^0$$

$$= V_s + c_1 + c_2$$

② $\dfrac{dV_c}{dt}(t=0) = ?$

$V \quad = L \dfrac{dI_L}{dt} \qquad \dfrac{dI_c}{}$

$$V_L = L\frac{dI_L}{dt} = L\frac{dI_C}{dt}$$

V can't be ∞, so current can't change instantaneously across an inductor

$$I_L(t<0) = I_L(t=0) = 0$$

$$= I_C(t=0) = C\frac{dV_C}{dt}(0)$$

$$\frac{dV_C}{dt}(t=0) = 0$$

$$\frac{dV_C}{dt} = \frac{d}{dt}\left(V_s + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right)\Bigg|_{t=0}$$

$$0 = c_1\lambda_1 + c_2\lambda_2$$

$$\Rightarrow \begin{bmatrix} V_C(0) \\ \frac{dV_C}{dt}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{I.C.'s}$$

$$V_s + c_1 + c_2 = 0$$

$$c_1 \lambda_1 + c_2 \lambda_2 = 0$$

$$\Rightarrow \quad c_1 = -5$$

$$c_2 = 1$$

$$\boxed{V_c(t) = 4 - 5e^{-10^{10}t} + e^{-(5\times 10^{10})t}}$$

overdamped

$$\lambda = \sigma + j\omega$$

$$e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}$$

underdamped