Dis 3C Notes

Tuesday, July 7, 2020 11:10 PM

Gram-Schmidt Process

- Inner Products
- Orthogonality
- G-S Procedure

Logistics: MT1 7/17?

⇒ Piazza post

I. Inner Products

Informally, an operation in a vector space that "multiplies" two vectors together.

① Linearity

a) $\langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

Additivity in 1st argument

b) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

Scaling in the 1st argument

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$$

2 $\overline{\langle x,y\rangle} = \langle y,x\rangle$

Conjugate Symmetry

3 $\langle x,x\rangle \geq 0$ with $\langle x,x\rangle = 0$ iff $\vec{x} = \vec{0}$

positive definiteness

a) dot product in $\mathbb{R}^N$

$$\langle \vec{x},\vec{y}\rangle = \vec{x}\cdot\vec{y} = \vec{x}^T\vec{y} = \sum x_i y_i$$

→ linear in both arguments

→ symmetric

2 Complex Inner Product

$$\langle \vec{u}, \vec{v} \rangle = u^T \overline{\vec{v}} = \overline{\vec{v}}^{*} \vec{u}$$

$$\overline{\vec{v}}^T \vec{u} = \sum_i \overline{v_i} u_i = \sum_i u_i \overline{v_i}$$

$$\vec{v}^{*} u$$

$*$ = conjugate transpose

Why?

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ j \end{bmatrix}$$

But not the $\vec{0}$ vector!

$$\vec{x} \cdot \vec{x} = 1^2 + j^2 = 0$$

$$\vec{y} = \begin{bmatrix} j \\ j \end{bmatrix} \Rightarrow \vec{y} \cdot \vec{y} = j^2 + j^2 = -2 < 0$$

$\Rightarrow$ Fix by adding complex conjugate

c) Functions? $\longrightarrow$ can be vectors!

$$\langle u, v \rangle = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \overline{v(\theta)} \, d\theta$$

$\langle \vec{v}, \vec{v} \rangle \quad \frac{1}{2\pi} \int_0 \ldots$

$\Rightarrow$ Explore in HW3 !
(Q6)

related:
Fourier transform
(EE120)

II. Orthogonality

a) Definitions

orthogonal: $\vec{v} \perp \vec{u} \iff \langle \vec{v}, \vec{u} \rangle = 0$

- orthogonal set of vecs: $\{ \vec{v}_1 \ldots \vec{v}_n \}$

$$\langle \vec{v}_i, \vec{v}_j \rangle = \begin{cases} \neq 0 & i = j \\ 0 & i \neq j \end{cases}$$

(All pairs of distinct vectors are orthogonal)

- orthonormal set of vectors: orthogonal set where $\langle v_i, v_i \rangle$

set of vectors, where $\langle v_i, v_i \rangle$
$= \|v_i\|^2$
$= 1$

$\Rightarrow$ orthogonal set of vecs with unit length

$$\langle \vec{v}_i, \vec{v}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}$$

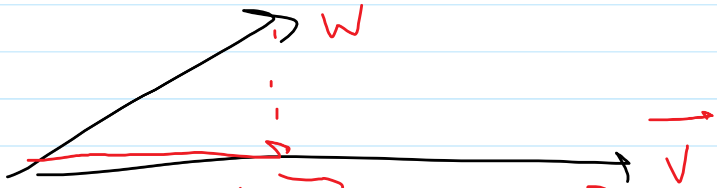
b) Projection (Orthogonal Projection)

$$\boxed{\operatorname{proj}_{\vec{v}} \vec{w} = \frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} = \frac{\langle \vec{w}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}}$$

How to remember expression?

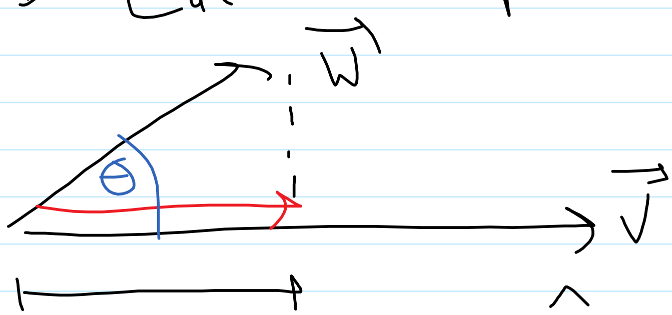
- $\operatorname{proj}_{\vec{v}} \vec{w}$ should point in same direction as $\vec{v}$
- $\vec{w}$ (vector being projected) only shows up once in expression

$\Rightarrow \vec{w}$



$$\mathrm{proj}_{\vec v}\vec w = \frac{\langle \vec w, \vec v\rangle}{\langle \vec v, \vec v\rangle}\vec v$$

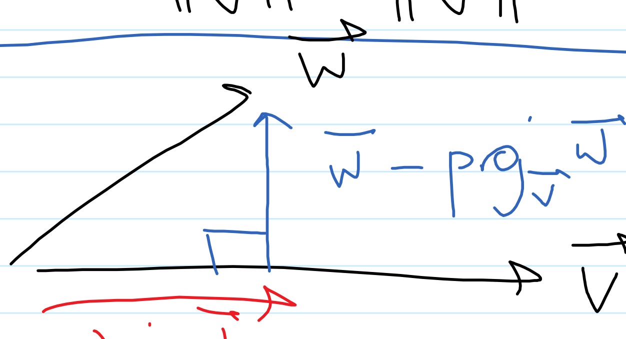
2D Euclidean space: (specific case, but helpful for trying to "derive" formula)



$$\|\vec w\|\cos\theta \times \hat v = \|\vec w\|\cos\theta \frac{\vec v}{\|\vec v\|}$$

$$\frac{\vec w \cdot \vec v}{\|\vec v\|} = \frac{\|\vec w\|\|\vec v\|\cos\theta}{\|\vec v\|}$$

$$\mathrm{proj}_{\vec v}\vec w = \frac{\vec w \cdot \vec v}{\|\vec v\|}\frac{\vec v}{\|\vec v\|} = \frac{\langle \vec w, \vec v\rangle}{\langle \vec v, \vec v\rangle}\vec v$$



$$\langle \vec{w} - \mathrm{proj}_{\vec{v}} \vec{w}, \vec{v} \rangle$$

$$= \langle \vec{w}, \vec{v} \rangle - \langle \mathrm{proj}_{\vec{v}} \vec{w}, \vec{v} \rangle$$

$$= \langle \vec{w}, \vec{v} \rangle - \left\langle \frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}, \vec{v} \right\rangle$$

$$= \langle \vec{w}, \vec{v} \rangle - \frac{\langle \vec{w}, \vec{v} \rangle}{\cancel{\langle \vec{v}, \vec{v} \rangle}} \cancel{\langle \vec{v}, \vec{v} \rangle}$$

$$= 0$$

c) Properties of Orthonormal Bases

- Consider orthogonal basis

$$\{ \vec{v}_1 \ \ldots \ \vec{v}_n \}$$

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n$$

$$\langle \vec{x}, \vec{v}_j \rangle = \left\langle \sum c_i \vec{v}_i , \vec{v}_j \right\rangle$$

all terms equal to 0 unless $i = j$

$$\langle \vec{x}, \vec{v}_j \rangle = c_j \langle \vec{v}_j, \vec{v}_j \rangle$$

$$\boxed{c_j = \frac{\langle \vec{x}, \vec{v}_j \rangle}{\langle \vec{v}_j, \vec{v}_j \rangle}}$$

Fast way to calculate coordinates in orthogonal basis $\vec{x}_b = B^{-1}\vec{x}_e$

- Orthonormal Bases

$$\langle \vec{v}_j, \vec{v}_j \rangle = 1 = \|\vec{v}_j\|^2$$

$$\boxed{c_j = \langle \vec{x}, \vec{v}_j \rangle}$$

Even faster!

← Orthogonal Matrix

A square matrix $U \in \mathbb{R}^{n \times n}$

s.t. the columns form an orthonormal basis

It can be shown that $U^T = U^{-1}$, i.e.,

$$U^T U = U U^T = I$$

- Unitary Matrix

$\Rightarrow$ complex analogue of orthogonal matrix

A square matrix $U \in \mathbb{C}^{n \times n}$ s.t. the columns form an orthonormal basis

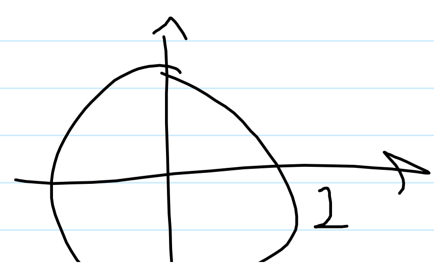
$$U^* = U^{-1}$$

(* here means conjugate transpose, aka adjoint)

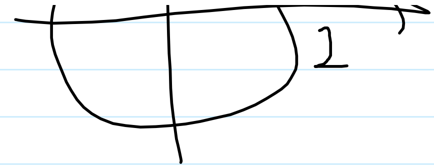
Lots of interesting properties, which you will see in HW3 (Q4)

One interesting property:

$$\boxed{|\det U| = 1}$$



$$\boxed{|\det U| = 1}$$



$$\det \mathbb{I} = \det U^* U$$

$$\det AB = \det A \det B = \det B \det A = \det BA$$

$$= \det U^* \det U$$

$$\det U^T = \det U$$

$$\Longrightarrow \det U^* = \overline{(\det U^T)} = \overline{\det U}$$

$$z\bar{z} = \overline{\det U} \det U$$

$$= |z|^2 \qquad = |\det U|^2 = \det \mathbb{I} = 1$$

$$\therefore |\det U| = 1$$

If U is orthogonal, $\det U \in \mathbb{R}$

then $\Longrightarrow \det U = \pm 1$

Consider orthogonal matrices!

$$U^T = U^{-1}$$

Rotations!

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(\theta)^T$$

$$\det R = \cos^2\theta + \sin^2\theta = 1$$

det 1 orthogonal matrices correspond to rotations!

$\Longrightarrow$ Subgroup of group of orthogonal transformations $O(n)$ that can be represented by $n \times n$ orthogonal matrices

$\Longrightarrow$ Subgroup where det $= 1$: special orthogonal group $SO(n)$

In physics, very important!

In physics, very important:

classical mechanics $SO(3)$

$\Longrightarrow$ "3D rotation group"

But what about $\det = -1$?

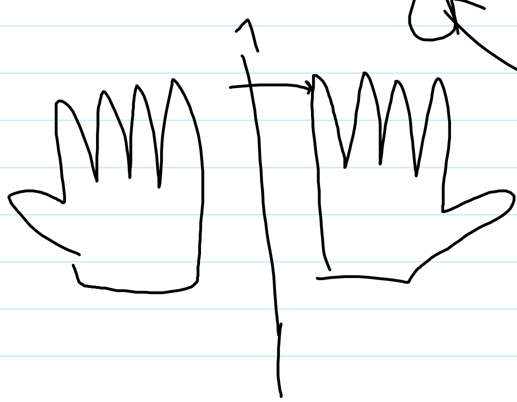
$\Longrightarrow$ reflections

$$\det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -1$$

$\Longrightarrow$ "parity" in physics and chemistry, i.e., flip the sign of one coordinate

$\Longrightarrow$ test of symmetry, e.g., chirality in chemistry

"handedness"



For unitary matrices $\Longrightarrow$ unitary group $U(n)$

$\det U = 1 \Longrightarrow$ special unitary

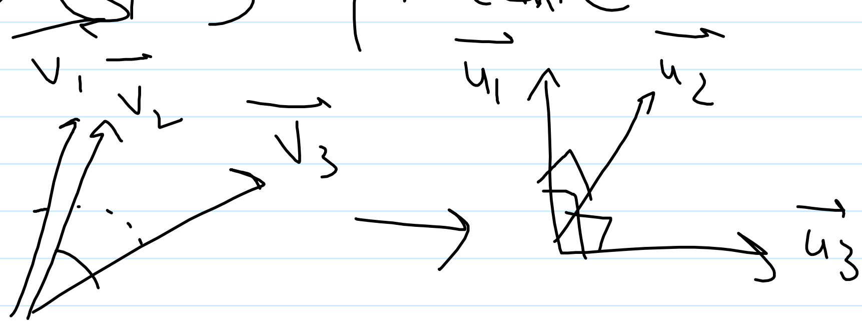
$\det U = 1 \Longrightarrow$ special unitary group $SU(n)$

group theory: abstract algebra?
(not sure) Math 113?

III. Gram-Schmidt

$\longrightarrow$ how do we make orthonormal sets of vectors?

$\longrightarrow$ G-S procedure

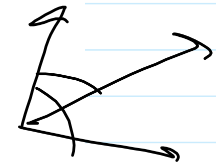


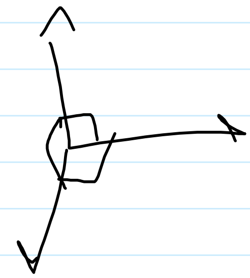
Dis 3C Worksheet

Tuesday, July 7, 2020 11:11 PM

3 Gram-Schmidt Algorithm

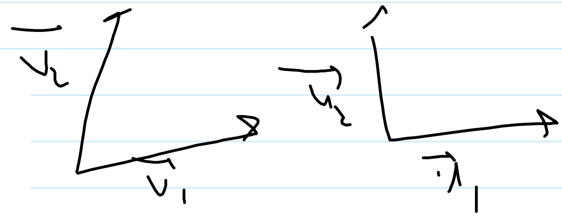
~~Let's apply Gram-Schmidt orthonormalization to a~~ set of three linearly independent vectors ~~$\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$~~. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \to \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$





a) Find unit vector $\cancel{\vec{q}_1}$ $\vec{u}_1$ such that span($\cancel{\{\vec{q}_1\}}$ $\vec{u}_1$) = span($\cancel{\{\vec{s}_1\}}$ $\vec{v}_1$).

$$\boxed{\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}}$$



b) Given $\cancel{\vec{q}_1}$ $\vec{u}_1$ from the previous step, find $\cancel{\vec{q}_2}$ $\vec{u}_2$ such that span($\cancel{\{\vec{q}_1, \vec{q}_2\}}$ $\{\vec{u}_1, \vec{u}_2\}$) = span($\{\vec{s}_1, \vec{s}_2\}$) $\{\vec{v}_1, \vec{v}_2\}$ and $\cancel{\vec{q}_2}$ $\vec{u}$ is orthogonal to $\cancel{\vec{q}_1}$ $\vec{u}_1$.

$$\vec{z}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}\vec{v}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1\rangle \vec{u}_1}{\cancel{\langle \vec{u}_1, \vec{u}_1\rangle}}$$

$$= \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1\rangle \vec{u}_1$$

$$\vec{u}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|}$$

c) Now given $\cancel{\vec{q}_1}$ $\vec{u}_1$ and $\cancel{\vec{q}_2}$ $\vec{u}_2$ in the previous steps, find $\cancel{\vec{q}_3}$ $\vec{u}_3$ such that span($\cancel{\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}}$ $\vec{u}_1, \vec{u}_2, \vec{u}_3$) = span($\cancel{\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}}$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$).

$$\vec{z} = \vec{v}_3 - \text{proj}\,\vec{v}_3 - \text{proj}\,\vec{v}_3$$

$$\vec{z}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1} \vec{v}_3 - \text{proj}_{\vec{u}_2} \vec{v}_3$$

$$= \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2$$

$$\vec{u}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|}$$

d) Let's extend this algorithm to n linearly independent vectors. That is, given an input $\{\vec{s}_1, \dots, \vec{s}_n\}$, write the algorithm to calculate the orthonormal set of vectors $\{\vec{q}_1, \dots, \vec{q}_n\}$, where $\text{span}(\{\vec{s}_1, \dots, \vec{s}_n\}) = \text{span}(\{\vec{q}_1, \dots, \vec{q}_n\})$. *Hint: How would you calculate the i^{th} vector, $\vec{q}_i$?*

Input: $\{\vec{v}_1 \dots \vec{v}_n\}$ (lin ind)

Output: $\{\vec{u}_1 \dots \vec{u}_n\}$ (orthonormal)

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

semicolons (I'm bad at writing)

for $i = 2;\ i \leq n;\ i{+}{+}$

$$\vec{z}_i = \vec{v}_i - \sum^{i-1} \langle \vec{v}_i, \vec{u}_i \rangle \vec{u}_i$$

$$j = 1$$

$$\vec{u}_1' = \frac{\vec{z}_i}{\|\vec{z}_i\|}$$

4 The Order of Gram-Schmidt

a) If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the set of vectors

$$\left\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\right\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\} \tag{4}$$

Perform Gram-Schmidt on these vectors first in the order $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\vec{z}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} - \left(\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}\right)\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\vec{u}_2 = \vec{z}_2 / \|\vec{z}_2\| = \vec{z}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\vec{z}_3 = \vec{v}_3 - \mathrm{proj}_{\vec{u}_1} \vec{v}_3 - \mathrm{proj}_{\vec{u}_2} \vec{v}_3$$

$$\vec{z_3} = \vec{v_3} - \text{proj}_{\vec{u_1}} \vec{v_3} - \text{proj}_{\vec{u_2}} \vec{v_3}$$

$$= \vec{v_3} - \langle \vec{v_3}, \vec{u_1} \rangle \vec{u_1} - \langle \vec{v_3}, \vec{u_2} \rangle \vec{u_2}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{u_3} = \vec{z_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

b) Now perform Gram-Schmidt on these vectors in the order $\vec{v}_3,\ \vec{v}_2,\ \vec{v}_1$. Do you get the same result?

$$\vec{z}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\vec{u}_3 = \vec{z}_3 / \|z_3\| = \begin{pmatrix}1\\1\\1\end{pmatrix}\frac{1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\vec{z}_2 = \vec{v}_2 - \operatorname{proj}_{\vec{u}_3}\vec{v}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} - \langle \vec{v}_2, \vec{u}_3\rangle \vec{u}_3$$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} - \left(\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}\right)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{3}(2)\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}$$

$$\vec{u}_2 = \begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}\frac{1}{\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{4}{9}}} = \sqrt{\frac{3}{2}}\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}$$

$$\vec{z}_1 = \vec{v}_1 - \operatorname{proj}_{\vec{u}_3}\vec{v}_1 - \operatorname{proj}_{\vec{u}_2}\vec{v}_1$$

$$= \vec{v}_1 - \langle \vec{v}_1, \vec{u}_3\rangle \vec{u}_3 - \langle \vec{v}_1, \vec{u}_2\rangle \vec{u}_2$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \left(\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$- \left(\sqrt{\frac{3}{2}}\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\sqrt{\frac{3}{2}}\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\ \cdots\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\1\\ \cdots\end{pmatrix} - \frac{3}{\cdots}\cdot\frac{1}{\cdots}\begin{pmatrix}1/3\\1/3\\ \cdots\end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{2}\cdot\frac{1}{3}\begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

$$\vec{u_3} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}} = \sqrt{2}\begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$

$$\boxed{\left\{\vec{u_1}, \vec{u_2}, \vec{u_3}\right\} = \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \sqrt{\frac{3}{2}}\begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}, \sqrt{2}\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \right\}}$$

Spans same space, but different basis!