Dis 3D Notes

Thursday, July 9, 2020 10:17 AM

Self-Adjoint / Hermitian Matrices

* Complex Inner Products
* Adjoints
* Fundamental Matrix Subspaces

Worksheet

* Self-Adjoint Matrices
  $\implies$ Spectral Theorem
* Fund. Thm. of Linear Algebra

$$A = \underbrace{U}_{\text{left singular vecs}} \Sigma \overbrace{V^*}^{\text{right singular vecs}}$$

I Complex Inner Products (Recap)

$$\boxed{\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \overline{\vec{y}} = \vec{y}^* \vec{x} = \sum x_i \overline{y_i}}$$

You can check these satisfy inner product properties:

① Linearity in 1st Argument

a) Additivity:

$$\langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$$

b) Scaling:

$$\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle$$

② Conjugate Symmetry

$$\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$$

③ Positive-Definiteness

$$\langle \vec{x}, \vec{x} \rangle \geq 0 \quad \text{with } \vec{x} = \vec{0} \iff \langle \vec{x}, \vec{x} \rangle = 0$$

Some notes on complex inner product:

$$\langle \vec{x}, \vec{y}+\vec{z} \rangle = (\vec{y}+\vec{z})^* \vec{x}$$

$$= \vec{y}^* \vec{x} + \vec{z}^* \vec{x}$$

$$= \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$$

Happens to be additive in both args

$$\langle \vec{x}, c\vec{y} \rangle = c^* \vec{y}^* \vec{x} = \bar{c}\, \vec{y}^* \vec{x}$$

$= \bar{c}\langle x, y\rangle$

But scaling only in 1st arg

$$\boxed{\text{Norm: } \|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v}\rangle}}$$

II. Adjoints

Technical definition: Consider $f: U \to V$

(lin. map b/w 2 vector spaces U and V)

$\Rightarrow$ Then the adjoint of f, f^* is defined as follows:

$\rightarrow$ $f^*: V \longrightarrow U$

$\langle f(\vec{u}), \vec{v}\rangle = \langle \vec{u}, f^*(\vec{v})\rangle$

f^* has to be a unique linear map that satisfies these properties

Very abstract! :(

$\Longrightarrow$ In matrix/vector notation, this means conjugate transpose i.e $V^* = \overline{V}^T$

means "conjugate transpose", i.e. $v^* = \bar{v}^T$

How to connect to abstract definition?

$A \in \mathbb{R}^{m \times n}$

$$m\underbrace{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}_{n} \; n\begin{pmatrix} \\ \\ \\ \end{pmatrix} = m\begin{pmatrix} \\ \\ \end{pmatrix}$$

A maps $\vec{v} \in \mathbb{R}^n$ to $\vec{w} \in \mathbb{R}^m$

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$U \quad \mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \quad V$$

Then transpose is $A^T \in \mathbb{R}^{n \times m}$

$$n\underbrace{\begin{pmatrix} & & \\ & & \end{pmatrix}}_{m}$$

A^T maps $\vec{w} \in \mathbb{R}^m$ to $\vec{v} \in \mathbb{R}^n$

$$V \quad \mathbb{R}^m \longrightarrow \mathbb{R}^n \quad U$$

Adjoint adds complex conjugation property to match complex inner product property

$$\Rightarrow \langle A\vec{u}, \vec{v} \rangle \stackrel{?}{=} \langle \vec{u}, A^* \vec{v} \rangle$$

$$\hookrightarrow \vec{v}^* A\vec{u} = (\vec{v}^* A)\vec{u}$$

$$= \langle \vec{u}, (\vec{v}^* A)^* \rangle$$

$$= \langle \vec{u}, A^* \vec{v}^{**} \rangle$$

$$= \langle \vec{u}, A^* \vec{v} \rangle$$

Compare to $\langle f(\vec{u}), \vec{v} \rangle = \langle \vec{u}, f^*(\vec{v}) \rangle$

b) Self-Adjointness / Hermiticity

$$f^* = f$$

e.g. $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \longleftarrow$ symmetric matrix

$$\boxed{\begin{array}{lll} \text{Symmetric:} & A^T = A & \text{(real-valued)} \\ \text{Hermitian:} & A^* = A & \text{(complex analogue)} \end{array}}$$

EX: Pauli Matrices and Quantum Computing

(C191)

$\Longrightarrow$ operators in QM have to be Hermitian if they correspond to observable values, b/c eigenvalues are what is observed and have to be $\underline{\underline{\text{real}}}$

(proved soon!)

2×2 Hermitian matrices:

$$\begin{pmatrix} a+b & \underline{c-di} \\ \underline{c+di} & a-b \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$+ d\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= a\bar{\underline{I}} + b\sigma_z + c\sigma_x + d\sigma_y$$

$aI + b\sigma_z + c\sigma_x + d\sigma_y$

Pauli matrices

$\Rightarrow \{I, \sigma_x, \sigma_y, \sigma_z\}$ form a basis for all 2×2 Hermitian matrices

In QC, basic unit of info known as a qubit, which is a "two-level system"

—— $|1\rangle$

—— $|0\rangle$

Essentially, state can be represented by unit vectors in $\mathbb{C}^2 \Rightarrow \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$

Then a single qubit transformation, if we would like to map one qubit state to another qubit state, should follow $f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$\Rightarrow$ 2×2 matrix!

like Pauli matrices ...

$\Rightarrow$ Pauli matrices represent basic

Single qubit transformations

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow$ maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

"0" "1"

maps $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

"1" "0"

$\Rightarrow$ "NOT" gate

σ_z : phase gate

σ_y : combination of "NOT" and phase change

III. Fundamental Matrix Subspaces

column space : span {columns of A}

$= C(A)$

row space : span {rows of A}

$= R(A) = C(A^T)$

null space : $\left\{ \begin{array}{l} \text{set of all vectors} \\ \vec{x} : A\vec{x} = \vec{0} \end{array} \right\} = Null(A)$

left null : (" ") " " / " "

Space : $\{ \vec{y} : A^T \vec{y} = \vec{0} \} = \mathrm{Null}(A')$

* dimension: size of the basis for a vector space $\mathbb{V}$ (# basis vectors)

* rank: dimension of $C(A)$ (sometimes called column rank, with $\dim R(A) \rightarrow$ row rank)

Note: common way to compute rank is to row reduce and count the # "pivots"

$$\begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

pivots

$\Rightarrow$ $\boxed{\text{row rank} = \text{column rank}}$

IV. Spectral Theorem

Why do we care?

$\rightarrow$ Identifies a class of matrix/ . . .

operators*** that can always be diagonalized***

Diagonalization makes a lot of computations easier:

EX: $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$ possible coupled set of diff eq

$\frac{d\vec{z}(t)}{dt} = \Lambda\vec{z}(t)$ uncoupled!

$\vec{z}(t) = V^{-1}\vec{x}(t) \longleftarrow A = V\Lambda V^{-1}$

EX: matrix exponentiation

$e^{\begin{bmatrix} & \\ & \end{bmatrix}} = ???$

$$e^{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}} = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}$$

Dis 3D Worksheet

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1 Spectral Theorem

For a complex $n \times n$ Hermitian matrix A,

a) All eigenvalues of A are real.

b) A has n linearly independent eigenvectors $\in \mathbb{C}^n$.

c) A has orthogonal eigenvectors, i.e., $A = V\Lambda V^{-1} = V\Lambda V^*$, where Λ is a diagonal matrix w/ real eigenvalues and V is a unitary matrix. We say that A is orthogonally diagonalizable.

Recall that a matrix A is Hermitian if $A = A^*$. Furthermore, if A is of the form B^*B for some arbitrary matrix B, all of its eigenvalues are non-negative, i.e., $\lambda \geq 0$.

a) Prove the following: All eigenvalues of a Hermitian matrix A are real.
Hint: Let $(\lambda, \vec{v})$ be an eigenvalue/vector pair and use the definition of an eigenvalue to show that $\lambda\langle\vec{v}, \vec{v}\rangle = \overline{\lambda}\langle\vec{v}, \vec{v}\rangle$.

$$z = \overline{z} \Leftrightarrow z \in \mathbb{R}$$

$$A\vec{v} = \lambda\vec{v}$$

$$\vec{v}^* A\vec{v} = \vec{v}^*\lambda\vec{v}$$

But note: $A = A^*$

LHS: $v^*Av = (v^*A^*)\vec{v} = (A\vec{v})^*\vec{v}$
$= (\lambda\vec{v})^*\vec{v} = \overline{\lambda}\,\vec{v}^*\vec{v}$

RHS: $\vec{v}^*\lambda\vec{v} = \lambda\,\vec{v}^*\vec{v}$

Equate the two sides:

$$\overline{\lambda}\,\cancel{(\vec{v}^*\vec{v})} = \lambda\,\cancel{(\vec{v}^*\vec{v})}$$

$$\Rightarrow \overline{\lambda} = \lambda \Longrightarrow \lambda \in \mathbb{R}$$

Alternatively, just using def'n of Hermiticity + inner products

$$\langle A\vec{v}, \vec{v}\rangle = \langle\vec{v}, A^*\vec{v}\rangle$$

$$\lambda\langle\vec{v}, \vec{v}\rangle = \langle\vec{v}, A\vec{v}\rangle = \overline{\lambda}\langle\vec{v}, \vec{v}\rangle$$

$\lambda \langle \vec{v}, \vec{v} \rangle \quad = \langle \vec{v}, A\vec{v} \rangle = \lambda \langle \vec{v}, \vec{v} \rangle$

b) Prove the following: For any Hermitian matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Hint: Use the definition of an eigenvalue to show that $\lambda_1 \langle \vec{v}_1, \vec{v}_2 \rangle = \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle$.

Orthogonality: $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$

$\Longrightarrow$ Show that $a \langle \vec{v}_1, \vec{v}_2 \rangle = b \langle \vec{v}_1, \vec{v}_2 \rangle$ where $a \neq b$ (e.g. $3x = 5x \Longrightarrow x = 0$)

$A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ where $\lambda_1 \neq \lambda_2$

$\langle A\vec{v}_1, \vec{v}_2 \rangle = \langle \lambda_1 \vec{v}_1, \vec{v}_2 \rangle = \lambda_1 \langle \vec{v}_1, \vec{v}_2 \rangle$

But from def'n of Hermiticity:

$\langle A\vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_1, A^* \vec{v}_2 \rangle = \langle \vec{v}_1, A\vec{v}_2 \rangle$

$= \langle \vec{v}_1, \lambda_2 \vec{v}_2 \rangle = \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle$

But $\overline{\lambda_2} = \lambda_2$ (from part (a))

$\lambda_1 \langle \vec{v}_1, \vec{v}_2 \rangle = \lambda_2 \langle \vec{v}_1, \vec{v}_2 \rangle$

$\Longrightarrow \langle \vec{v}_1, \vec{v}_2 \rangle = 0 \Longrightarrow \vec{v}_1 \perp \vec{v}_2$

c) Prove the following: For any matrix A, A^*A is Hermitian and only has non-negative eigenvalues.

i) $(A^*A)^* = A^* A^{**} = A^*A$ ✓

(Hermiticity)

ii) $v^* \left(A^*A\vec{v} = \lambda \vec{v} \right)$ Eigenvalue Eqn

$(\vec{v}^* A^*) A \vec{v} = \lambda \vec{v}^* \vec{v}$

$$(\vec{v}^* A^*) A\vec{v} = \lambda \vec{v}^* \vec{v}$$

$$\langle A\vec{v}, A\vec{v}\rangle = \lambda \langle \vec{v}, \vec{v}\rangle$$

$$\Rightarrow \lambda = \frac{\langle A\vec{v}, A\vec{v}\rangle}{\langle \vec{v}, \vec{v}\rangle} = \frac{\|A\vec{v}\|^2}{\|\vec{v}\|^2} \geq 0$$

2 Fundamental Theorem of Linear Algebra

a) Let $\vec{v}$ be an eigenvector of nonzero eigenvalue of A^*A. Show that $\vec{v} \in \mathrm{Col}(A^*)$.

$$(A^*A)\vec{v} = \lambda \vec{v} \longleftarrow \lambda \neq 0$$

$$A^*(A\vec{v}) = A^*\vec{y} = \lambda \vec{v}$$

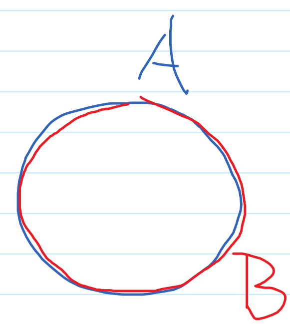
By definition, $\lambda \vec{v} \in A^*$

(scaling preserved)

$\Rightarrow$ vector space closed under scalar mult., by def'n

b) Show that the two subspaces $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*A)$ are equal.

Hint: you can show $A = B$ if $A \subseteq B$ and $B \subseteq A$



① Suppose $\vec{x} \in \mathrm{Null}(A)$

$$A\vec{x} = \vec{0}$$

$$(A^*A)\vec{x} = \vec{0} \quad \therefore \vec{x} \in \mathrm{Null}(A^*A)$$

$(A^*A)\vec{x} = 0$

② $\vec{x} \in \text{Null}(A^*A) \overset{?}{\Longrightarrow} \vec{x} \in \text{Null}(A)$

$A^*A\vec{x} = \vec{0}$

$x^*A^*A\vec{x} = \vec{0}$ (By positive-definiteness)

$\langle A\vec{x}, A\vec{x} \rangle = \vec{0}$

$\|A\vec{x}\|^2 = 0$ $\longrightarrow A\vec{x} = \vec{0}$

$\therefore \vec{x} \in \text{Null}(A)$

c) Let $\vec{u}$ be an eigenvector of eigenvalue 0 of A^*A. Show that $\vec{u} \in \text{Nul}(A)$.

$(A^*A)\vec{u} = \lambda\vec{u} = \vec{0}$ $(\lambda = 0)$

$\therefore \vec{u} \in \text{Null}(A^*A)$

From (b), $\therefore$ $\vec{u} \in \text{Null}(A)$

rank A^* = rank $A = k$

d) If A is a $m \times n$ matrix of rank k what are the dimensions of $\text{Col}(A^*)$ and $\text{Nul}(A)$?

Row space

row rank = dim $R(A)$ = column rank = dim $\text{Col}(A)$

$\Rightarrow$ dim $\text{Col}(A^*)$ = rank A = k

$$\Longrightarrow \boxed{\dim \operatorname{Col}(A^*) = \operatorname{rank} A = k}$$

$$\boxed{\dim \operatorname{Null}(A) = n - k}$$

by the rank-nullity theorem

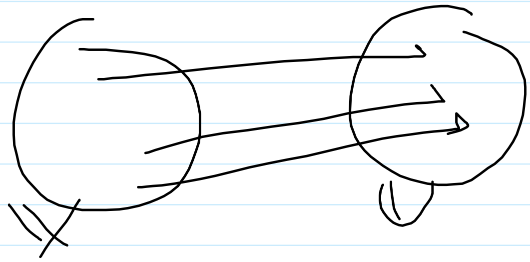
* Rank-Nullity Theorem:

For an $m \times n$ matrix $A : V \longrightarrow U$

$$\dim \operatorname{Null}(A) + \operatorname{rank} A = \dim V$$

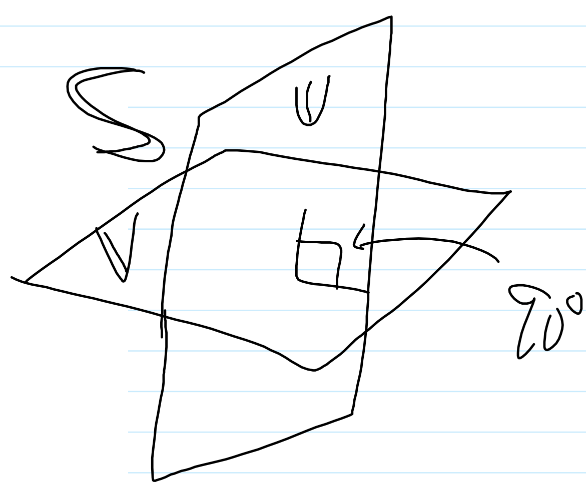
For example, $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\dim V = n$



Then $\underbrace{\dim \operatorname{Null}(A) + \operatorname{rank} A = n}$

e) Use parts (a)-(d) to show that $\operatorname{Col}(A^*)$ is the orthogonal complement of $\operatorname{Nul}(A)$.
Use the spectral theorem on the matrix A^*~~A to create an orthonormal eigenbasis of~~ $\mathbb{C}^n$



$$\operatorname{Null}(A) \perp \operatorname{Col}(A^*)$$

$$A^*A = (n \times m)(m \times n)$$
$$= n \times n \text{ matrix}$$

① By spectral theorem, since A^*A is

① By spectral theorem, since A^*A is Hermitian, we can create a full basis of orthonormal eigenvectors

$$\underbrace{\vec{v}_1 \;\dots\; \vec{v}_j}_{\lambda_i \neq 0} \quad \underbrace{\vec{v}_{j+1} \;\dots\; \vec{v}_n}_{\lambda_i = 0}$$

We will show that the eigenvectors for nonzero eigenvalues of A^*A correspond to an $\perp$ basis for $\text{Col}(A^*)$, while the eigenvectors for zero eigenvals correspond to an $\perp$ basis for $\text{Null}(A)$

② The eigenvectors of A^*A w/ $\lambda = 0$ form an orthonormal basis for $\text{Null}(A^*A)$ by construction.

From part (b), $\underline{\text{Null}(A) = \text{Null}(A^*A)}$

Thus, the eigenvecs of A^*A for $\lambda = 0$ also form an $\perp$ basis for $\text{Null}(A)$

③ Since $\text{Null}(A) = \text{Null}(A^*A)$,

$$\dim \text{Null}(A) = \dim \text{Null}(A^*A)$$

$n-k$ from part (d)

Then, using the rank-nullity theorem on $n\times n$ matrix A^*A:

$$\text{rank } A^*A + \dim \text{Null}(A^*A) = n$$

$$\Longrightarrow \text{rank } A^*A + (n-k) = n$$

$$\Longrightarrow \text{rank } A^*A = k = \text{rank } A = \text{rank } A^*$$

Then $\text{Col}(A^*)$ and $\text{Col}(A^*A)$ have the same dimension!

Note that from part (a), the eigenvecs of A^*A for nonzero eigenvals are part of $C(A^*)$.

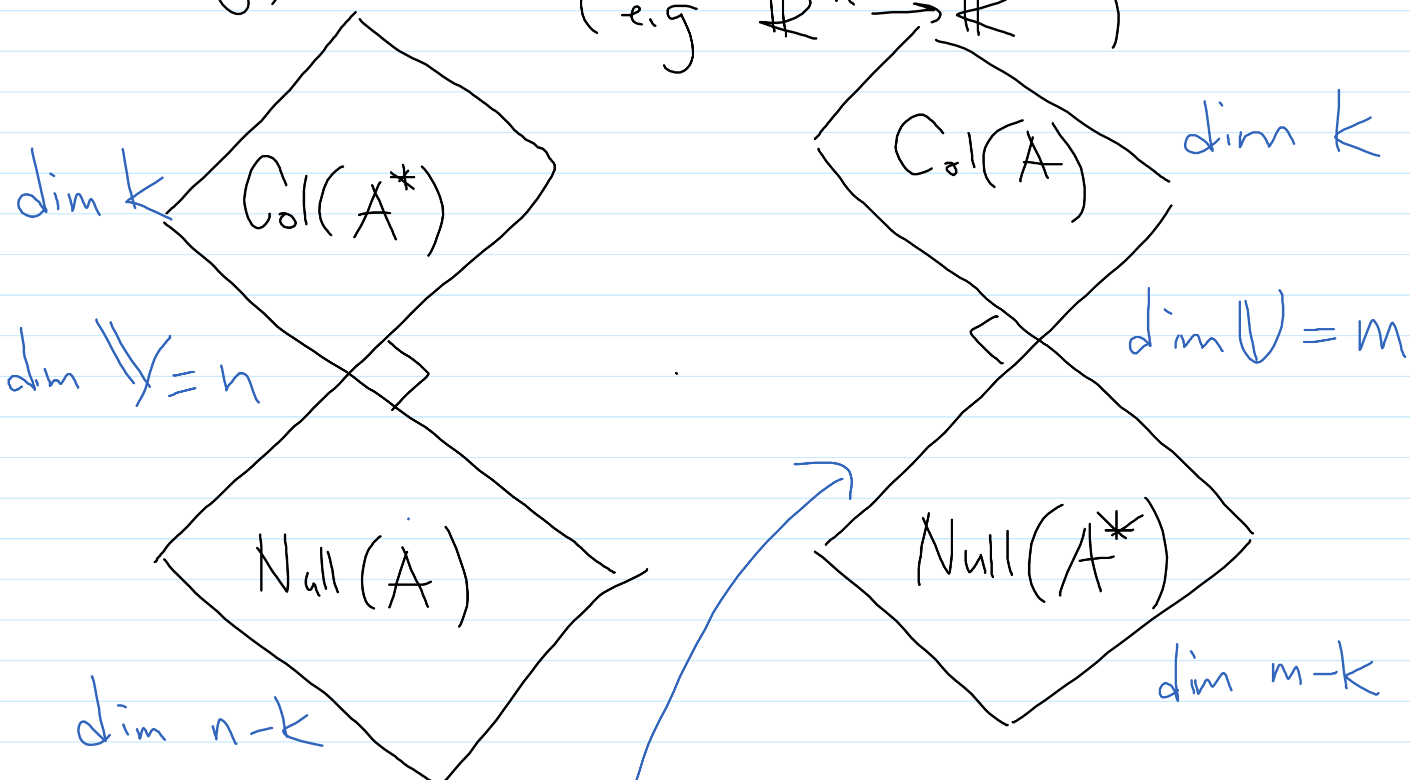
Since $\dim \text{Col}(A^*) = \dim \text{Col}(A^*A) = k$

the k eigenvectors of A^*A s.t. $\lambda \neq 0$ can be used to form an $\perp$ basis for $\text{Col}(A^*)$

We have split the n eigenvectors of A^*A into k eigenvecs ($\lambda \neq 0$) that form a basis for $\text{Col}(A^*)$ and $n-k$ eigenvecs ($\lambda = 0$) that form a basis for $\text{Null}(A)$. By the spectral theorem, these basis sets are orthogonal.

$\therefore \ \mathrm{Col}(A^*) \perp \mathrm{Null}(A)$

Visually, for $A : V \to U$ (e.g. $\mathbb{R}^n \to \mathbb{R}^m$)



Note: A similar result can be proved for $\mathrm{Col}(A)$ and $\mathrm{Null}(A^*)$, i.e. they are orthogonal complements (except this time $\mathrm{Col}(A)$ and $\mathrm{Null}(A^*)$ are subspaces of U, so

$\dim \mathrm{Col}(A) + \dim \mathrm{Null}(A^*) = \dim U$)

$\dim \mathrm{Col}(A) + \dim \mathrm{Null}(A^*) = \dim W$)