Dis 4A Notes

Sunday, July 12, 2020 8:57 PM

SVD I

* Introduction
* Calculating the SVD
* SVD & Fundamental Matrix Subspaces

→ Worksheet!

Helpful resource: https://medium.com/@jonathan_hui/machine-learning-singular-value-decomposition-svd-principal-component-analysis-pca-1d45e885e491

Logistics:

* MT 1 on Friday 7/17, 5-7pm (PST)
* HW4 due 7/21 (on SVD, PCA, linearization)
* HW Party moved to F, M 4-6pm

This week: F HW party is at 1-3pm

Review sessions: Tu, Wed 8-9pm

Tu: ckts

Wed: linear algebra

I. Introduction to SVD

$n \times n$: $A = V \Lambda V^{-1}$

$n \times n$ matrices: $A = V \Lambda V^{-1}$

If $A^* = A$ (Hermitian),

$$A = \underset{\text{unitary matrix}}{V} \underset{\text{diagonal (real-valued)}}{\Lambda} V^*$$

We factorized A into a product of interesting matrices

$\Longrightarrow$ Would be great if we could extend factorization to any $m \times n$ matrix

* SVD *

Viewpoint 1: SVD as a matrix factorization

$$A = U \Sigma V^*$$

$$A = U \Sigma V$$

$m \times n \qquad m \times m \qquad m \times n \qquad n \times n$

U, V are square unitary matrices:

$$U^* U = I_{m \times m}$$

$$V^* V = I_{n \times n}$$

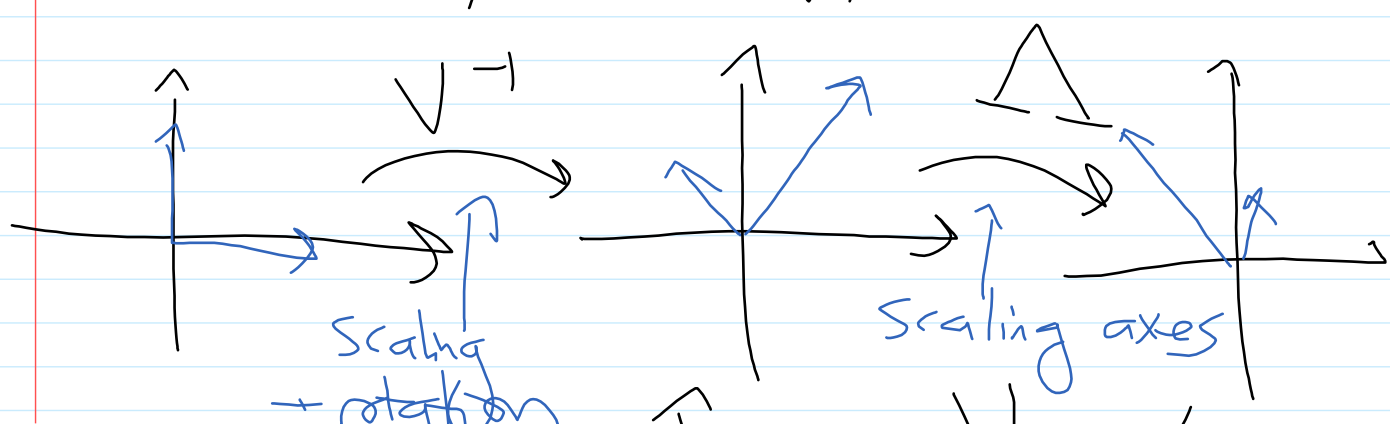
Σ is a rectangular diagonal matrix w/ diagonal entries $\sigma_i = \Sigma_{ii} \geq 0$

($\sigma_i > 0$ for $1 \leq i \leq r$ where $r = \operatorname{rank} A$)

Interpretation?

$$A = V \Lambda V^{-1}$$

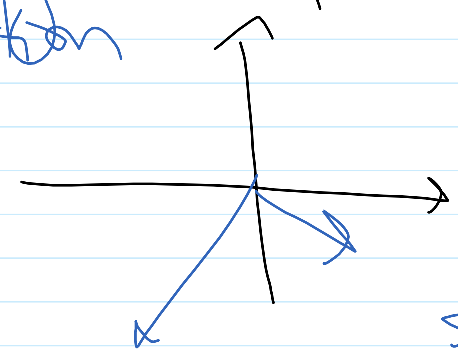
Similar! $A\vec{x} = V \Lambda V^{-1} \vec{x}$



scaling + rotation

scaling axes

V



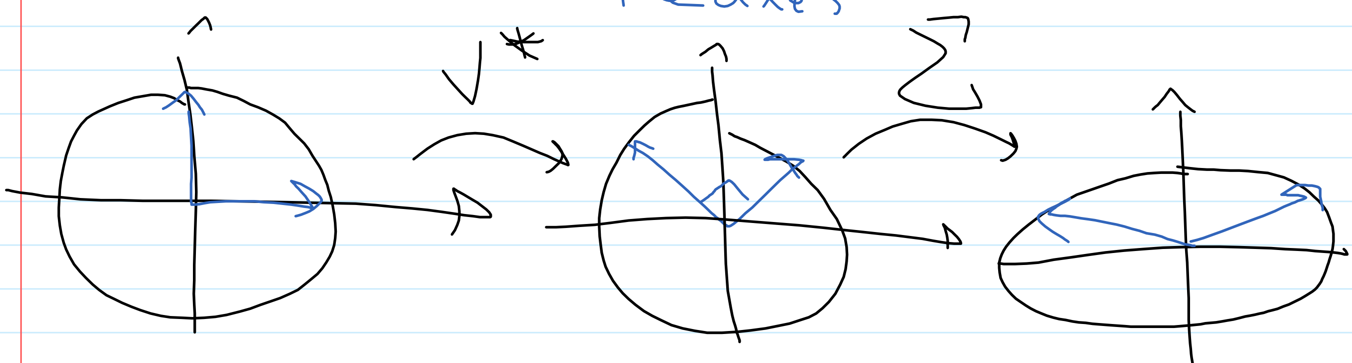
scaling + rotation

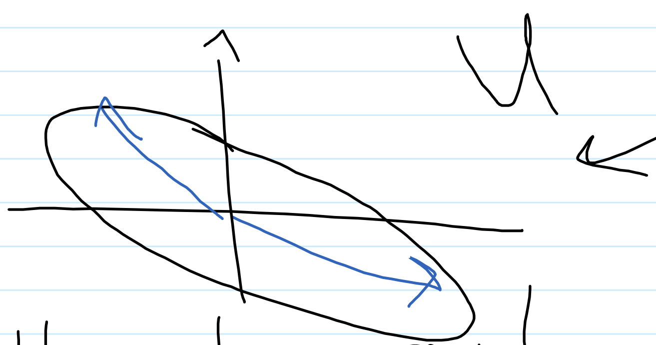
SVD: $A = U \Sigma V^*$

U: unitary matrix — rotation

Σ: rect diag — scaling the axes

V^*: unitary matrix — rotation





More on this tomorrow!

(Geometric interpretation)

Viewpoint 2: SVD as a sum of rank 1 matrices

$$A = U \Sigma V^* = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \left(\begin{matrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m \end{matrix} \middle| \; 0 \right) \times \begin{pmatrix} \vdash & v_1^* & \dashv \\ \vdash & v_2^* & \dashv \\ & \vdots & \\ \vdash & v_n^* & \dashv \end{pmatrix}$$

(Assumed $n > m$ but end result will be the same for $n < m$)

$$= \begin{pmatrix} | & & | \\ \sigma_1 \vec{u}_1 & \cdots & \sigma_m \vec{u}_m \\ | & & | \end{pmatrix} \begin{pmatrix} \vdash & v_1^* & \dashv \\ \vdash & \cdots & \dashv \\ & \vdots & \\ \vdash & v_n^* & \dashv \end{pmatrix}$$

"outer products"

$$\vec{x}\vec{y}^T = \underset{m \times 1}{\begin{pmatrix} \\ \\ \end{pmatrix}} \underset{1 \times n}{\begin{pmatrix} & \end{pmatrix}} = \begin{matrix} m \times n \\ \text{rank } 1 \\ \text{matrix} \end{matrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \begin{pmatrix} y_1^* & \cdots & y_n^* \end{pmatrix} = \begin{pmatrix} x_1 \vec{y}^* \\ x_2 \vec{y}^* \\ \vdots \\ x_m \vec{y}^* \end{pmatrix}$$

rows all lin. dependent!

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^* + \sigma_2 \vec{u}_2 \vec{v}_2^* + \ldots + \sigma_m \vec{u}_m \vec{v}_m^*$$

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^* + \sigma_2 \vec{u}_2 \vec{v}_2^* + \dots + \sigma_m \vec{u}_m \vec{v}_m^*$$

Recall that $\sigma_i = 0$ for $i > r$

$$\boxed{A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^*}$$ Compact SVD

Sum of r rank-1 matrices

Benefit of this viewpoint?

$\Longrightarrow$ low rank matrix approx

$A \in \mathbb{C}^{m \times n} \Longrightarrow mn$ entries

$A = \sum \sigma_i u_i v_i^* \Longrightarrow r(m+n)$ entries

$\mathbb{C}^m$ $\quad n$

Nice if
$r(m+n) \ll mn$

Can further approximate / ~~compress~~ compress to a good degree if only a few singular values are significant:

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\tilde{r}} >> \sigma_{\tilde{r}+1} \dots \sigma_r$$

only take these values!

"truncated SVD"

What is the "best" low rank matrix approx?

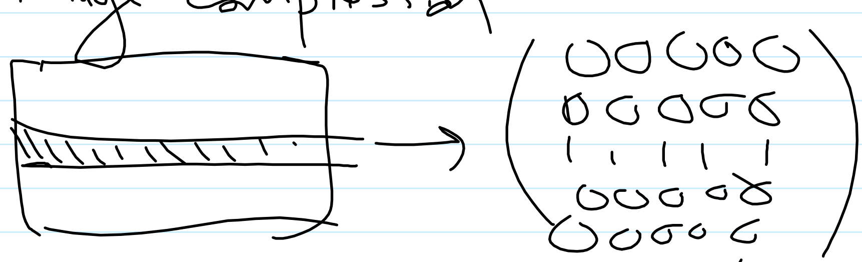
$\Longrightarrow$ SVD

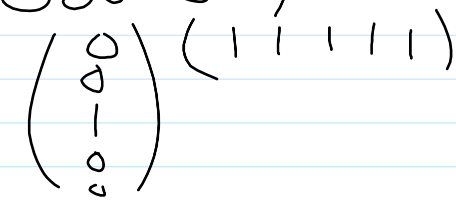
Eckart – Young – (Mirsky) Thm

— won't prove here

— Wikipedia's proof incorrect?

EX: image compression





Viewpoint 3: Abstract SVD.

(HW3, Q6)

Matrix Form: $A = U\Sigma V^*$

$$V^*V = I$$

$$U^*U = I$$

$$\Rightarrow AV = U\Sigma$$

$$\Rightarrow \boxed{A\vec{v_i} = \sigma_i \vec{u_i}}$$

In general, SVD of a linear map

$$f: V \longrightarrow U$$

is comprised of:

1) orthonormal basis for V

(right singular vecs)

2) " " for U

(left singular vecs)

3) $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0$

(r positive singular values)

where $f(\vec{v_i}) = \sigma_i u_i$

II. Constructing the SVD

$\Rightarrow$ a lot of inspiration from spectral theorem!

A^*A: — Hermitian
— eigenvalues are real
— orthogonally diagonalizable
— corollary: for $B = A^*A$, $\lambda \geq 0$ (dis 3D)

$\Rightarrow$ construct SVD from here?

Using $A^*A \in \mathbb{C}^{n \times n}$

(1) Find eigenvals λ_i of A^*A.

Order them s.t.:

$$\lambda_1 \geq \cdots \geq \lambda_r > 0$$

$$\lambda_{r+1} = \cdots = \lambda_n = 0$$

(2) Find orthonormal eigenvecs of A^*A for $i = 1 \ldots r$

$$\Rightarrow (A^*A)\vec{v_i} = \lambda_i \vec{v_i}$$
$$(i = 1 \ldots r)$$

$\Rightarrow$ right singular vectors of A^*A

3. Set $\sigma_i = \sqrt{\lambda_i}$ for $i = 1 \ldots r$

4. Left singular vecs:

Note $A\vec{v_i} = \sigma_i \vec{u_i}$

$$\Rightarrow \boxed{\vec{u_i} = \frac{A\vec{v_i}}{\sigma_i}} \quad (i = 1 \ldots r)$$

5. Complete the $\perp$ bases for U and V

$\hookrightarrow$ m vectors $\qquad \hookrightarrow$ n vectors

by end

(Gram–Schmidt)

Note: proof that $\vec{u_i}$ are $\perp$

$$\vec{u_i} = \frac{A\vec{v_i}}{\quad}$$

$$\vec{u}_j = \frac{A\vec{v}_j}{\sigma_j}$$

$$\langle \vec{u}_i, \vec{u}_j \rangle \stackrel{?}{=} \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\| $$

$$u_j^* u_i = \frac{\vec{v}_j^{\,*}(A^* A)\vec{v}_i}{\sigma_i \sigma_j} = \frac{\lambda_i}{\sigma_i \sigma_j} \vec{v}_j^{\,*} \vec{v}_i$$

$$= \frac{\sigma_i^2}{\sigma_i \sigma_j} \langle v_i, v_j \rangle$$

$$= \frac{\sigma_i}{\sigma_j} \langle v_i, v_j \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

Something interesting...

Essentially solving:
$$\begin{cases} A = U \Sigma V^* \\ U^* U = I \\ V^* V = I \end{cases}$$

$$A^* = V \Sigma^* U^*$$

$$A^*A = V\Sigma^* \cancel{U^*U} \Sigma V^*$$
$$= V\hat{\Sigma}^2 V^*$$

$$\begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix}$$

diagonalization!

$V \longrightarrow$ eigenvecs of A^*A are right singular vectors

$$AA^* = U\Sigma V^* V \Sigma^* U^*$$
$$= U\hat{\Sigma}^2 U^* \qquad \hat{\Sigma}^2 = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_m^2 \end{pmatrix}$$

$\Longrightarrow$ eigenvecs of AA^* are left singular vecs of A

Suggests procedure where we calculate SVD from AA^* (WS)

III. SVD relation to matrix subspaces

$$A\vec{v_i} = \sigma_i u_i$$

① Consider $\sigma_i = 0$ for $i > r$

$$\Longrightarrow A\vec{v_i} = \vec{0}$$

Right singular vecs corresponding to vanishing singular values span $Null(A)$

② Consider $i = 1 \dots r$, $\sigma_i \neq 0$

$$A\vec{v_i} = \sigma_i \vec{u_i}$$

Recall $\{\vec{v_i}\}$ span $\mathbb{C}^N$!

Then left singular vectors corresponding to nonzero σ_i span $Col(A)$

What about $Null(A^*)$, $Col(A^*)$?

$$A^* = (U\Sigma V^*)^*$$

$= V \Sigma^* U^*$

$= V \Sigma^T U^* \implies A^* U = V \Sigma^T$

$A^* \vec{u_i} = \sigma_i \vec{v_i}$

① left singular vectors of A corresponding to $\sigma_i = 0$ span $Null(A^*)$

② Right singular vecs of A corresponding to $\sigma_i \neq 0$ span $Col(A^*)$

Summary!

$$AA^* : \overbrace{\underbrace{\vec{u}_1 \ldots \vec{u}_r}_{\text{span } Col(A)}, \underbrace{\vec{u}_{r+1} \ldots \vec{u}_m}_{\text{span } N(A^*)}}^{\text{span } \mathbb{C}^m}$$

$$A^*A : \underbrace{\vec{v}_1 \ldots \vec{v}_r}_{\text{span } Col(A^*)}, \underbrace{\vec{v}_{r+1} \ldots \vec{v}_n}_{\text{span } Null(A)}$$

$$\underbrace{\text{Span Col}(A^*) \qquad \text{span Null}(A)}_{\text{Span } \mathbb{C}^n}$$

Dis 4A Worksheet

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1 SVD and Fundamental Subspaces

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

a) Find the SVD of A (compact form is fine).

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

$$A^*A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 18, \quad \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 0, \quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\sigma_1 = \sqrt{18} = 3\sqrt{2}$$

$$\sigma_2 = 0$$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

$$\boxed{A = \sigma_1 \vec{u}_1 \vec{v}_1^* = 3\sqrt{2}\begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}$$

b) Find the rank of A.

$$\boxed{\text{rank } A = 1}$$

$\hookrightarrow$ only 1 nonzero σ_i

c) Find a basis for the kernel (or nullspace) of A.

$$A v_i = \sigma_i u_i$$

$$\Rightarrow A\vec{v_2} = 0\vec{u_2} = \vec{0}$$

$$\boxed{\text{Null}(A) = \text{span}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\}}$$

d) Find a basis for the range (or columnspace) of A.

$$A\vec{v_i} = \sigma_i\vec{u_i}$$

$\Rightarrow$ nonzero σ_i

$$\Rightarrow \boxed{\text{Col}(A) = \text{span}\{\vec{u_1}\} = \text{span}\left\{\begin{bmatrix}1/3\\-2/3\\2/3\end{bmatrix}\right\}}$$

e) Repeat parts (a) - (d), but instead, create the SVD of $B = A^*$. What are the relationships between the answers for A and the answers for $B = A^*$?

$$B = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$(A = U\Sigma V^*)^* \Rightarrow B = A^* = V\Sigma^* U^*$$

$$\boxed{B = 3\sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}}$$

$$\boxed{\text{rank } A^* = 1}$$

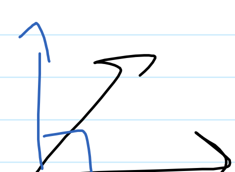
$$\boxed{\text{Col}(A^*) = \text{span}\left\{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\right\}}$$

Nullspace of A^*?

$\lambda_2 = \lambda_3 = 0$ (for AA^*)

$$\begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}\vec{v} = \vec{0}$$

Try $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$



Gram-Schmidt!

$$\vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} / \sqrt{2}$$

$$\vec{z_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \left\langle \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\rangle \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\boxed{\text{Null}(A) = \text{span}\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{3\sqrt{2}} \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \right\}}$$

2 Understanding the SVD

We can compute the SVD for a wide matrix A with dimension m x n where $n > m$ using A^*A with the method described above. However, when doing so you may realize that A^*A is much larger than AA^* for such wide matrices. This makes it more efficient to find the eigenvalues for AA^*. In this question we will explore how to compute the SVD using AA^* instead of A^*A.

a) What are the dimensions of AA^* and A^*A.

$A \in \mathbb{C}^{m \times n}$

AA^*: $m \times m$

A^*A: $n \times n$

b) Given that the $A = U\Sigma V^*$, find a symbolic expression for AA^*.

$$AA^* = (U\Sigma \cancel{V^*)(V} \Sigma^* U^*)$$

$$I$$

$$= U\Sigma\Sigma^* U^*$$

$$\boxed{B = AA^* = U\hat{\Sigma}^2 U^*}$$

$$\hat{\Sigma}^2 = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_m^2 \end{pmatrix}$$

c) Using the solution to the previous part explain how to find U and Σ from AA^*.

$AA^* = U\hat{\Sigma}^2 U^*$ diagonalization!

$\Longrightarrow$ find eigenvectors of AA^* $\{\vec{u}_i\}$ and eigenvalues $\{\lambda_i\}$

$\Longrightarrow$ $\sigma_i = \sqrt{\lambda_i}$

d) Now that we have found the singular values σ_i and the corresponding vectors $\vec{u}_i$ in the matrix U, devise a way to find the corresponding vectors $\vec{v}_i$ in matrix V.

$$\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i} \longrightarrow \boxed{\vec{v}_i = \frac{A^*\vec{u}_i}{\sigma_i}}$$

$$A^* = (U\Sigma_i V^*)^* = V\Sigma_i^* U^*$$

$$A^* = (U\Sigma V^*)^* = V\Sigma^* U^*$$

$$A^* U = V\Sigma^*$$

$$\Rightarrow A^* \vec{u}_i = \overline{\sigma_i} \vec{v}_i = \sigma_i \vec{v}_i$$

$$(\sigma_i \in \mathbb{R})$$

e) Now we have a way to find the vectors $\vec{v}_i$ in matrix V, verify that they are orthonormal.

$$\left.\begin{aligned} \vec{v}_i &= \frac{A^* \vec{u}_i}{\sigma_i} \\ \vec{v}_j &= \frac{A^* \vec{u}_j}{\sigma_j} \end{aligned}\right\} \longrightarrow \langle \vec{v}_i, \vec{v}_j \rangle = \vec{v}_j^{\,*} \vec{v}_i$$

$$= \frac{\vec{u}_j^{\,*} A A^* \vec{u}_i}{\sigma_i \sigma_j}$$

$$= \frac{\vec{u}_j^{\,*} \lambda_i \vec{u}_i}{\sigma_i \sigma_j}$$

Note $\lambda_i = \sigma_i^2$

$$= \frac{\sigma_i^2}{\sigma_i \sigma_j} u_j^* u_i$$

$$= \frac{\sigma_i}{\sigma_j} \langle u_i, u_j \rangle$$

if $i = j$:

$$\langle v_i, v_j \rangle = \frac{\sigma_i}{\sigma_i} \langle u_i, u_i \rangle = 1$$

if $i \neq j$

$$\langle v_i, v_j \rangle = \frac{\sigma_i}{\sigma_j} \cancel{\langle u_i, u_j \rangle}^{0} = 0$$

$$= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

f) Now that we have found $\vec{v}_i$ you may notice that we only have $m < n$ vectors of dimension n. This is not enough for a basis. How would you complete the m vectors to form an orthonormal basis?

Start: $\{\vec{v}_1 \dots \vec{v}_m\} \longrightarrow$ need n to span start: $\{\vec{v}_1 \ldots v_m\}$ $\longrightarrow$ need n to span n-dim space!

Use G-S procedure!

- Specifically, need to keep $\vec{v}_1 \ldots \vec{v}_m$ for SVD, but $\vec{v}_{m+1} \ldots \vec{v}_n$ will all get mapped to $\vec{0}$ anyway so they can be arbitrary as long as $\{\vec{v}_1 \ldots \vec{v}_m, \vec{v}_{m+1} \ldots \vec{v}_n\}$ is an orthonormal set
- We also need to ensure that the set spans the whole space; so add the entire standard basis in $\{\vec{e}_1 \ldots \vec{e}_n\}$

Thus, we perform G-S on:

$$\{\vec{v}_1 \ldots \vec{v}_m, \vec{e}_1 \ldots \vec{e}_n\}$$

Any vector that is redundant will become $\vec{0}$ after G-S

(redundant = lin. dep. on the already orthonormalized vectors)

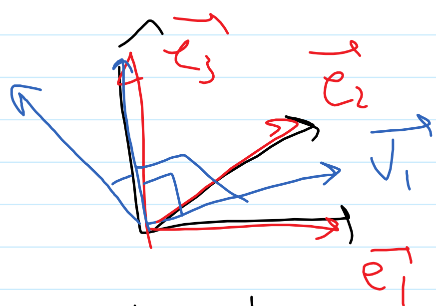
EXAMPLE:

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

$$\vec{z}_4 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\rangle \begin{pmatrix}1\\0\\0\end{pmatrix} - \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \begin{pmatrix}0\\1\\0\end{pmatrix} - \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \vec{0}$$



So our basis set becomes:

$$\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix} \right\}$$

remove this

In Summary:

1) Add std basis in so our spanning set becomes $\{\vec{v}_1 \dots \vec{v}_m, \vec{e}_1 \dots \vec{e}_n\}$

2) Orthonormalize w/ G-S, starting

with $\vec{e_i}$ (~~keep~~ $\vec{v_1} \dots \vec{v_n}$ untouched)

3) Remove the m zero vectors that result so you have a basis of n vectors to span an n-dim vector space!

g) Using the previous parts of this question and what you learned from lecture write out a procedure on how to find the SVD for any matrix.

1. Pick the smaller of A^*A and AA^*

$$A \in \mathbb{R}^{1000 \times 2}$$
$$A^*A \in \mathbb{R}^{2 \times 2}$$
$$AA^* \in \mathbb{R}^{1000 \times 1000}$$

2. Find the eigenvals and eigenvecs
   a) if A^*A: find $A^*A\vec{v_i} = \lambda_i \vec{v_i}$
   b) if AA^*: find $AA^*\vec{u_i} = \lambda_i \vec{u_i}$
3. $\sigma_i = \sqrt{\lambda_i}$
4. Complete U, V with G-S