Dis 4D Notes

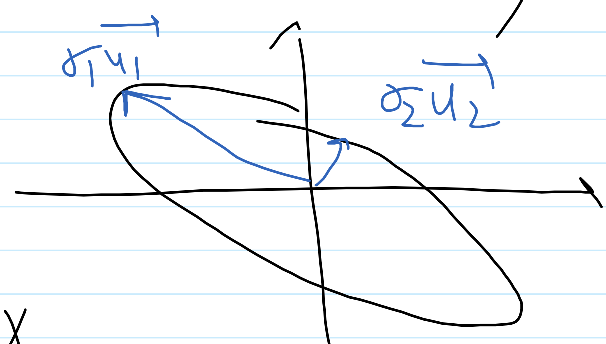
Thursday, July 16, 2020 12:06 PM

* Some Notes on PCA
* Review Worksheet
* Open Questioning

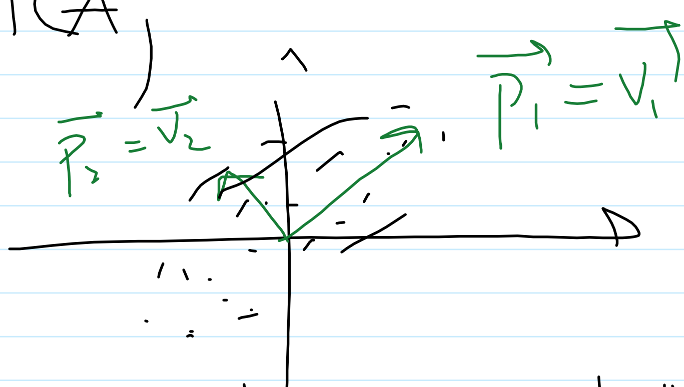
I. Notes on PCA

a) How come for SVD,

$AS =$



But for PCA,



In the two examples, we are talking about different situations.

- SVD example: concerned w/ the output

Take [illegible] transform it w/ A and

Take data, transform it w/ A, and ask "which output direction is the most stretched?"

- PCA example: concerned w/ the input optimization problem where we maximize the variance $\|\tilde{A}\vec{w}\|^2$ (variance along projection onto $\vec{w}$)

Take data represented in A, and we ask; "Which input direction would get stretched the most?"

b) Projecting data:

$$\tilde{A}\vec{w} = \begin{bmatrix} \vdash (x_1 - \bar{x})^T \dashv \\ \vdash (x_2 - \bar{x})^T \dashv \\ \vdots \\ \vdash (x_m - \bar{x})^T \dashv \end{bmatrix} \vec{w} = \begin{bmatrix} \langle x_1 - \bar{x}, w \rangle \\ \vdots \\ \langle x_m - \bar{x}, w \rangle \end{bmatrix}$$

Scalar projection along 1 vector $\vec{w}$

$$F = \tilde{A} P$$

projection onto the entire PCA basis P

$m \times n$ $\quad m \times n$ $\quad n \times n$

data becomes uncorrelated

$$S_F = \frac{1}{m} (\tilde{A} P)^T (\tilde{A} P)$$

$$= \frac{1}{m} P^T (\tilde{A}^T \tilde{A}) P$$

$$= \frac{1}{m} P^{-1} (\tilde{A}^T \tilde{A}) P$$

covariance matrix of F

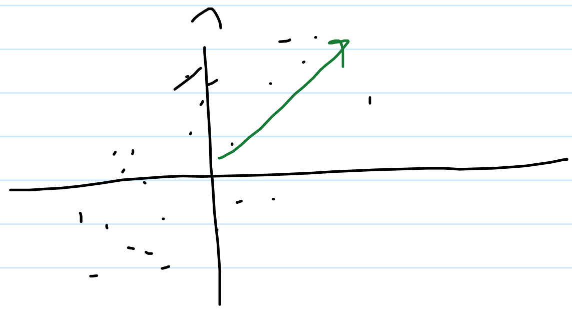
S_F is diagonal!

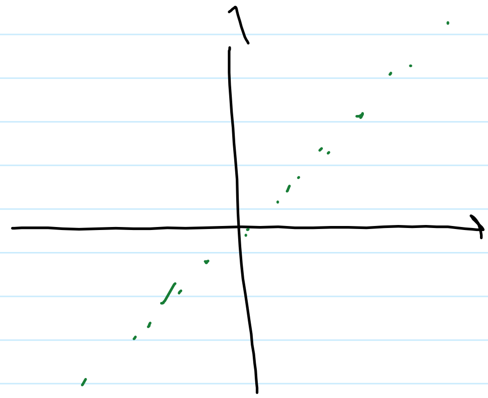
Dimensionality reduction:

$$\hat{F} = \tilde{A} \hat{P} \longleftarrow$$

project onto only 1st k principal components

$m \times k \qquad m \times n \qquad n \times k$





Don't think (?) this is the same as low rank matrix approx.

$$A = \sum \sigma_i u_i v_i^* \in \mathbb{C}^{m \times n}$$

$$\hat{F} = \hat{A}\hat{P} \in \mathbb{C}^{m \times k}$$

$$A \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_k \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \sum \sigma_i u_i v_i^* v_1 & \cdots & \sum \sigma_i u_i v_i^* v_k \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \sigma_1 u_1 & \cdots & \sigma_k u_k \\ | & & | \end{bmatrix}$$

$$= (\hat{U} \hat{\Sigma}) \longleftarrow m \times k$$

$$= (\hat{U}\hat{\Sigma}) \leftarrow m \times k \text{ truncated version of } U\Sigma$$

- Outerproduct

$$\vec{x}\vec{y}^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \begin{pmatrix} \vdash & \vec{y}^T & \dashv \end{pmatrix}$$

$$= \begin{pmatrix} \vdash & x_1 \vec{y}^T & \dashv \\ \vdash & x_2 \vec{y}^T & \dashv \\ & \vdots & \\ \vdash & x_m \vec{y}^T & \dashv \end{pmatrix}$$

- ~~Energy~~

$$P(t) = \frac{dU}{dt} = (IV)(t)$$

$$\int dU = \int P(t)\,dt = \int (IV)(t)\,dt$$

$$U = \int_0^\infty \frac{V_{DD}}{R} e^{-t/RC}\, V_{DD}\, dt$$

$$= \frac{V_{DD}^2}{} \frac{e^{-t/RC}}{} \Big|^\infty$$

$$= \frac{V_{DD}}{R} \quad \frac{e}{-1/RC}\Bigg|_0$$

$$= \frac{V_{DD}^2}{\cancel{R}} \cdot -\cancel{R}C\left(0 - 1\right)$$

$$= C V_{DD}^2$$

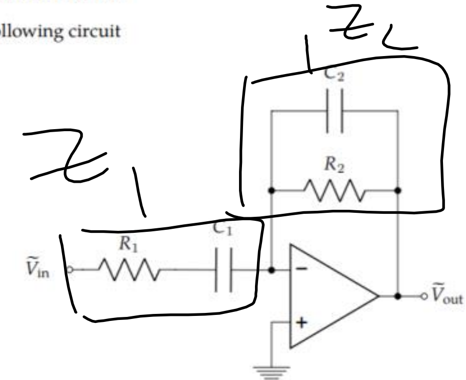
$$\underline{U_C = \frac{1}{2} C V^2}, \quad U_L = \underline{\frac{1}{2} L I^2}$$

Dis 4D Worksheet

Thursday, July 16, 2020 12:07 PM

1 Differentiator Circuit

Consider the following circuit



1 Transform to phasor domain

2 KCL, KVL, etc.

3 Find $\dfrac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$

1. What is the transfer function $H(j\omega)$?

$$H(j\omega) = -\frac{Z_2}{Z_1}$$

$$Z_2 = R_2 \parallel \frac{1}{j\omega C_2}$$

$$Z_1 = R_1 + \frac{1}{j\omega C_1}$$

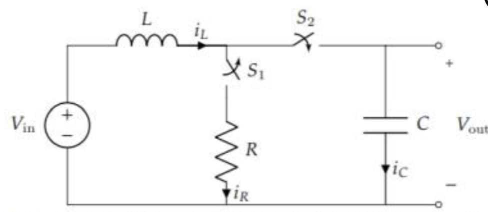
$$Z_2 = \frac{R_2 \frac{1}{j\omega C_2}}{R_2 + \frac{1}{j\omega C_2}} \cdot \frac{j\omega C_2}{j\omega C_2} = \frac{R_2}{1 + j\omega R_2 \underbrace{C_2}_{C_2}}$$

$$H(j\omega) = -\frac{Z_2}{Z_1} = \frac{\frac{R_2}{1+j\omega R_2 C_2}}{R_1 + \frac{1}{j\omega \cancel{C_1}}} \cdot \frac{j\omega C_1}{\cancel{j\omega C_1}}$$

$$\boxed{H(j\omega) = -\frac{j\omega R_2 C_1}{(1+j\omega R_2 C_2)(1+j\omega R_1 C_1)}}$$

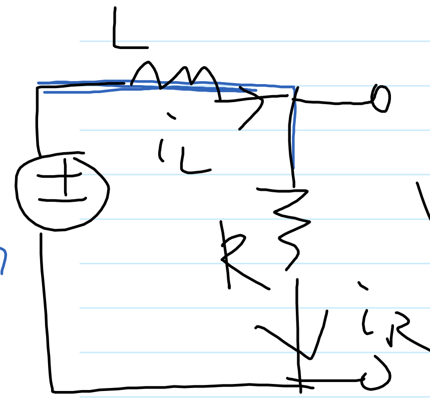
2 Parallel RLC

Consider the circuit shown below.



At $t < 0$, S_1 is on (short-circuited), and S_2 is off (open-circuited).
At $t \geq 0$, S_1 is off (open-circuited), and S_2 is on (short-circuited).

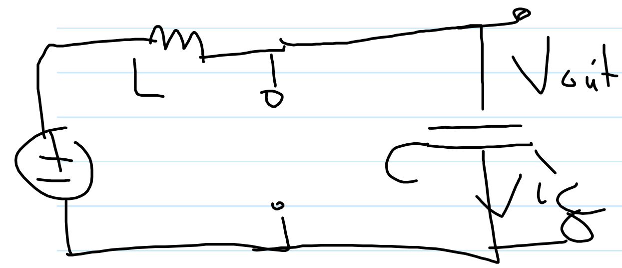
$t<0$



1. Right after the switches change state (i.e., at $t = 0$), what is the value of i_L?

steady state?

$t>0$



Phasors: $Z_c = \frac{1}{j\omega C}$

DC input, steady state: $Z_C \to \infty$
$(\omega = 0)$

OC at steady state

$Z_L = j\omega L$

DC input, steady state: $Z_L \to 0$
$(\omega = 0)$

SC at steady state

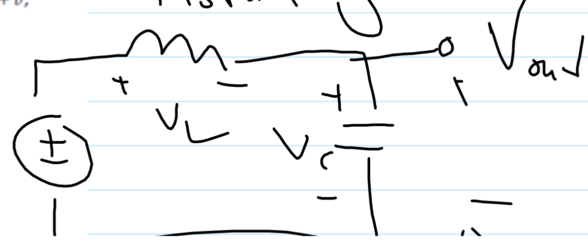
$i_L(t<0) = \frac{V_{in}}{R}$

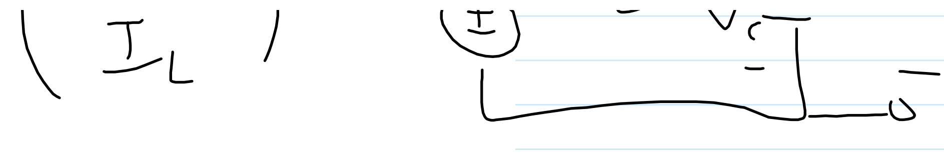
$\boxed{i_L(t=0) = \frac{V_{in}}{R}}$

$V_L = L\frac{d\bar{I}_L}{dt} \longrightarrow V_L$ can't be ∞, so I_L can't change instantly

2. Choosing the state variables as $\vec{x}(t) = \begin{bmatrix} V_{out}(t) \\ i_L(t) \end{bmatrix}$, derive the **A** matrix that captures the behavior of this circuit for $t \geq 0$ with the matrix differential equation $\frac{d\vec{x}(t)}{dt} = \mathbf{A}\vec{x}(t) + \vec{b}$, where $\vec{b}$ is a vector of constants.

$\vec{x}(t) = \begin{pmatrix} V_{out} \\ I_L \end{pmatrix}$





$V_c = V_{out}$

KVL: $V_{in} = V_L + V_{out} = L\frac{di_L}{dt} + V_{out}$

$$\Rightarrow \frac{di_L}{dt} = \frac{V_{in}}{L} - \frac{V_{out}}{L} \quad \text{DE 1}$$

$$i_L = i_C = C\frac{dV_C}{dt} = C\frac{dV_{out}}{dt}$$

$$\Rightarrow \frac{dV_{out}}{dt} = \frac{1}{C} i_L$$

$$\boxed{\frac{d\vec{x}}{dt} = \begin{bmatrix}\dot{V}_{out}\\ \dot{i}_L\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{C}\\ -\frac{1}{L} & 0\end{bmatrix}\begin{bmatrix}V_{out}\\ i_L\end{bmatrix} + \begin{bmatrix}0\\ +\frac{V_{in}}{L}\end{bmatrix}}$$

3. Assuming that $V_{\text{out}}(0) = 0\,\text{V}$, derive an expression for $V_{\text{out}}(t)$ for $t \geq 0$.

① Inhomog → homogeneous

$$\tilde{V}_{out} = V_{out} - V_{in}$$

$$\frac{d\tilde{V}_{out}}{dt} = \frac{dV_{out}}{dt}$$

$$L \rightarrow \frac{di_L}{dt} = \frac{V_{in} - V_{out}}{L} = \frac{-\tilde{V}_{out}}{L}$$

$$\frac{d\tilde{V}_{out}}{dt} = \frac{1}{C} i_L$$

$$\frac{di_L}{dt} = \frac{-\tilde{V}_{out}}{L}$$

$$\rightarrow \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} -\lambda & \frac{1}{C} \\ -\frac{1}{L} & -\lambda \end{bmatrix} = \lambda^2 + \frac{1}{LC} = 0$$

$$\lambda = \frac{\pm j}{\sqrt{LC}}$$

$$\tilde{V}_{out}(t) = V_{out}(t) - V_{in} : c_1 e^{\frac{+j}{\sqrt{LC}}t} + c_2 e^{\frac{-j}{\sqrt{LC}}t}$$

$$\Rightarrow V_{out}(t) = V_{in} + c_1 e^{\frac{j}{\sqrt{LC}}t} + c_2 e^{\frac{-j}{\sqrt{LC}}t}$$

$$V_{out}(0) = 0 = V_{in} + c_1 + c_2 \qquad (1)$$

$$\frac{d\vec{X}}{dt}(0) = A\vec{x}(0) = \begin{bmatrix} \frac{d}{dt}\left(c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\right) \\ \frac{d}{dt}\left(\qquad \right) \end{bmatrix}$$

$$\frac{dV_{out}}{dt}(0) = \frac{i_L(0)}{C} = c_1\lambda_1 + c_2\lambda_2$$

$$\frac{V_{in}}{RC} = (c_1 - c_2)\frac{j}{\sqrt{LC}} \qquad (2)$$

$$c_1 = V_{in}\left(\frac{1}{2j}\sqrt{\frac{L}{C}}\frac{1}{R} - \frac{1}{2}\right)$$

$$c_2 = V_{in}\left(-\frac{1}{2j}\sqrt{\frac{L}{C}}\frac{1}{R} - \frac{1}{2}\right)$$

$$V_{out}(t) = V_{in}\left(1 - \cos\left(\frac{1}{\sqrt{LC}}t\right)\right)$$
$$+ V_{in}\sqrt{\frac{L}{C}}\frac{1}{R}\sin\left(\frac{1}{\sqrt{LC}}t\right)$$

3 Diagonalizability and Invertibility

1. Given an example of a matrix A, or prove that no such example can exist.

a) • Can be diagonalized and is invertible.
b) • Cannot be diagonalized but is invertible.
c) • Can be diagonalized but is non-invertible.
d) • Cannot be diagonalized and is non-invertible.

a) $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}$$

$$= (\lambda - 1)^2 \Longrightarrow \lambda = 1, 1$$

$A - \lambda I$

$$\text{Null} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

c) $$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow$$ Hermitian, so diagonalizable

$$(1-\lambda)^2 - 1 = \lambda^2 - 2\lambda + 1 - 1$$
$$= \lambda(\lambda - 2)$$

$\to$ lin dep cols $\Longrightarrow$ not invertible

d) $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow \lambda = 0$$

$$\text{Null}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

4 Eigenvalue Decomposition and Singular Value Decomposition

We define Eigenvalue Decomposition as follows:

If a matrix $A \in \mathbb{R}^{n \times n}$ has n linearly independent eigenvectors $\vec{p}_1, \ldots, \vec{p}_n$ with eigenvalues $\lambda_i, \ldots, \lambda_n$, then we can write:

$$A = P \Lambda P^{-1}$$

Where columns of P consist of $\vec{p}_1, \ldots, \vec{p}_n$, and Λ is a diagonal matrix with diagonal entries $\lambda_i, \ldots, \lambda_n$.

Consider a matrix $A \in \mathbb{S}^n$, that is, $A = A^T \in \mathbb{R}^{n \times n}$. This is a symmetric matrix and has orthorgonal eigenvectors. Therefore its eigenvalue decomposition can be written as,

$$A = P \Lambda P^T$$

1. First, assume $\lambda_i \geq 0, \forall i$. Find the SVD of A.
2. Let one particular eigenvalue λ_j be negative, with the associated eigenvector being p_j. Succinctly,

$$A p_j = \lambda_j p_j \text{ with } \lambda_j < 0$$

We are still assuming that,

$$A = P \Lambda P^T$$

a) What is the singular value σ_j associated to λ_j?

b) What is the relationship between the left singular vector u_j, the right singular vector v_j and the eigenvector p_j?

$A = A^T$, $AA^T = A^TA = A^2$

$\implies \text{SVD}\{A\} \implies \vec{v}_i =$ eigenvecs of A^TA

$\vec{u}_i =$ eigenvecs of AA^T

Expect that $\vec{u}_i = \vec{v}_i$

$\hookrightarrow$ However, have to be careful about the eigenvalues: are they always nonnegative?

$$A^TA = (P \Lambda P^T)^T P \Lambda P^T$$

$= P\Lambda \cancel{P^T P} \Lambda P^T = P\Lambda^2 P^T$

$\sigma_i^2 = \lambda_i^2$, $V = P$

Here, λ_i refers to the eigenvals of A

if $\lambda_i \geq 0$, $\sigma_i = \lambda_i$

if $\lambda_i < 0$, $\sigma_i = |\lambda_i|$

Case 1: $\lambda_i \geq 0$, $\sigma_i = \lambda_i$

$$\Rightarrow \vec{u_i} = \frac{A\vec{v_i}}{\sigma_i} = \frac{\lambda_i \vec{v_i}}{\sigma_i} = \vec{v_i}$$

$$\therefore U = V = P$$

$$\Rightarrow A = U\Sigma V^T = V \Sigma V^T$$

And since $\sigma_i = \lambda_i$,

$$\boxed{A = V\Lambda V^T = P\Lambda P^T}$$

Thus, for $\underline{\underline{\lambda_i \geq 0}}$,

SVD is the $\underline{\underline{\text{same}}}$ as diagonalization

(for symmetric matrices)

b) What if λ_j is negative?

[i] $A = P\Lambda P^T$ (diag)

$A = U\Sigma V^T$ (SVD)

$A^TA = P\Lambda^2 P^T \longrightarrow$ eigenvecs are $\vec{p_i} = \vec{v_i}$, eigenvals are σ_i^2

$\Rightarrow \sigma_i^2 = \lambda_{i,A}^2 = \lambda_{i,A^TA}$

If $\lambda_j < 0$, σ_j still has to be > 0

$$\boxed{\sigma_j = -\lambda_j = |\lambda_j|}$$

[ii] $A = \sum \sigma_i u_i v_i^*$

To account for negative sign, throw it onto either $\vec{u_j}$ or $\vec{v_j}$

$\Rightarrow$ $\vec{v_j} = -\vec{p_j}$ and $\vec{u_j} = \vec{p_j}$

OR $\vec{v_j} = \vec{p_j}$ and $\vec{u_j} = -\vec{p_j}$

$$\boxed{\text{OR } \vec{v_j} = \vec{p_j} \quad \text{and} \quad u_j = -\vec{p_j}}$$