Dis 5A Notes

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Linearization

* State Space
* Equilibrium Points
* Linearization
  - Scalar case
  - Vector case (Jacobian)

→ Worksheet

I. State Space Models

- state variables: set of variables that fully represent the state of a dynamical system at any point in time

State vector $X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathcal{X}$ (state space)

$$\begin{bmatrix} \vdots \\ x_n(t) \end{bmatrix} \quad \text{(state space)}$$

inputs $\vec{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \in \mathcal{U}$ (control space)

— Continuous Systems

$$\boxed{\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), \vec{u}(t))}$$

vector-valued function w/ vector inputs

state $\vec{x} \in X$

inputs $\vec{u} \in \mathcal{U}$

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} f_1(\vec{x}, \vec{u}) \\ \vdots \\ f_n(\vec{x}, \vec{u}) \end{bmatrix}$$

not necessarily linear!

— If linear: matrix!

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + B\vec{u}(t)$$

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + B\vec{u}(t)$$

- Preview: discrete time

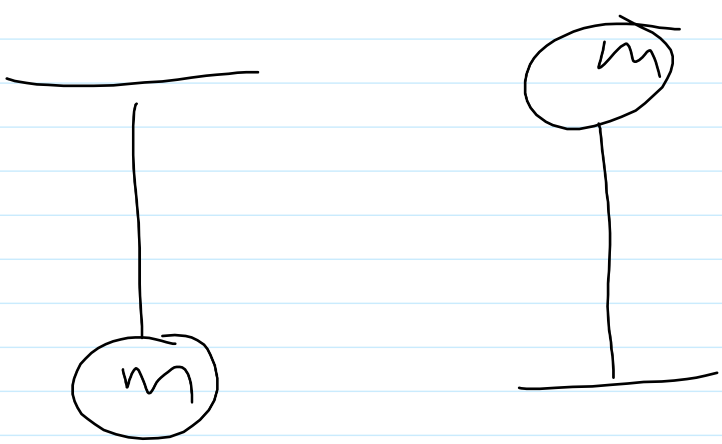
$$\frac{d\vec{x}}{dt} \longrightarrow \vec{x}[n+1]$$

$$\boxed{\vec{x}[n+1] = f(\vec{x}[n], \vec{u}[n])}$$

$$\boxed{\text{Linear: } \vec{x}[n+1] = A\vec{x}[n] + B\vec{u}[n]}$$

II. Equilibrium Points

Intuition: "state doesn't change in time"



No inputs: $\vec{u}(t) = 0$

$$\frac{\quad}{dt} = f(\vec{x}(t))$$

$$\Longrightarrow \frac{d\vec{x}^*}{dt} = f(\vec{x}^*) = 0$$

Constant solution!

Formally, $\boxed{\mathcal{X}_{eq} = \{\vec{x} \in \mathcal{X} \mid f(\vec{x}) = 0\}}$

No inputs

With inputs:

$$\boxed{\mathcal{X}_{eq}(\vec{u}) = \{\vec{x} \in \mathcal{X} \mid f(\vec{x}, \vec{u}) = 0\}}$$

- Linear Systems

$$\frac{d\vec{x}}{dt} = A\vec{x} + B\vec{u} = \vec{0}$$

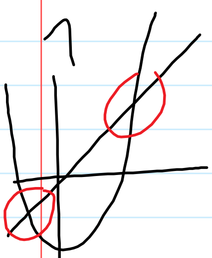
-

Solve for $\vec{x}$ in equation

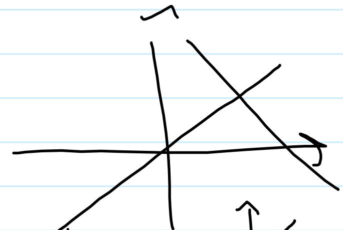
$$A\vec{x} = -B\vec{u}$$

Linear algebra question!

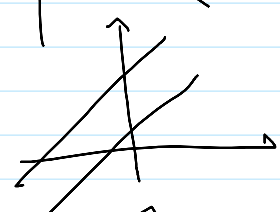
"Solve $A\vec{x} = \vec{y}$"



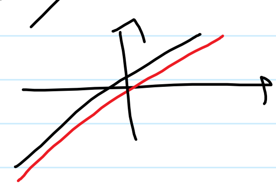
1) unique solution



2) no solution



3) infinite solutions



$\Rightarrow$ (1) A invertible

$$\underbrace{\vec{x} = -A^{-1}B\vec{u}}$$

(2) A is not invertible

a) infinite solutions $(\vec{x}_p + \vec{x}_h)$

if $B\vec{u} \in \mathrm{Col}(A)$

b) no solutions
if $B\vec{u} \notin Col(A)$

III. Linearization

a) scalar case

Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x^*)}{n!}(x-x^*)^n$$

$$= f(x^*) + \frac{f'(x^*)}{1!}(x-x^*)$$

$$+ \frac{f''(x^*)}{2!}(x-x^*)^2 + \ldots$$

"Taylor expansion about $x = x^*$"

$\Longrightarrow$ Linearization: just take the 1st 2 terms!

2 terms!

$$\boxed{f(x) \simeq f(x^*) + f'(x^*)(x - x^*)}$$

Alternate ways of saying same thing:

$$\delta f = f(x) - f(x^*)$$

$$\delta x = x - x^*$$

- $f(x) \simeq f(x^*) + f'(x^*)\,\delta x$

- $x - x^* = \delta x$, so $x = x^* + \delta x$

$\Rightarrow f(x^* + \delta x) = f(x^*) + f'(x^*)\,\delta x$

- $\delta f = f'(x^*)\,\delta x$

(HW4, Q5)

* What if f is a fn of many variables?

$$f(x_1 \dots x_n) = f(x_1^* \dots x_n^*) + \left.\frac{\partial f}{\partial x_1}\right|_{\vec{x}=\vec{x}^*} (x_1 - x_1^*)$$

$$+ \left.\frac{\partial f}{\partial x_2}\right|_{\vec{x}=\vec{x}^*} (x_2 - x_2^*)$$

$$+ \dots$$

$$\boxed{f(\vec{x}) = f(\vec{x}^*) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} (x_j - x_j^*)}$$

As matrix multiplication:

$$f(\vec{x}) = f(\vec{x}^*) + \left.\vec{\nabla} f(\vec{x})\right|_{\vec{x}=\vec{x}^*} (\vec{x} - \vec{x}^*)$$

$$= f(\vec{x}^*) + \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ \vdots \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ x_n - x_n^* \end{bmatrix}$$

b) Vector case and Jacobians

$$f(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{bmatrix}$$

$$f(\vec{x}) \approx f(\vec{x}^*) + \begin{bmatrix} \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j}(x_j - x_j^*) \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j}(x_j - x_j^*) \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

In the end,

$$\vec{f}(\vec{x}) \approx f(\vec{x}^*) + \nabla f \Big|_{\vec{x} = \vec{x}^*} (\vec{x} - \vec{x}^*)$$

$$\Big|_{\vec{x}=\vec{x}^*}$$

In the case of $f(\vec{x},\vec{u})$

$$\frac{d\vec{x}}{dt} = f(\vec{x},\vec{u})$$

$$\Longrightarrow \underbrace{\frac{d\vec{x}}{dt}}_{n\times 1} \approx \underbrace{f(\vec{x}^*,\vec{u}^*)}_{n\times 1} + \underbrace{\nabla_x f\Big|_{\vec{x}^*,\vec{u}^*}}_{n\times n} \underbrace{\delta\vec{x}}_{n\times 1} + \underbrace{\nabla_u f\Big|_{\vec{x}^*,\vec{u}^*}}_{n\times m} \underbrace{\delta\vec{u}}_{m\times 1}$$

$$\frac{d\vec{x}(t)}{dt} \approx f(\vec{x}^*,\vec{u}^*) + A\vec{x} + B\vec{u}$$

$$\frac{d\vec{x}(t)}{dt} = f(\vec{x},\vec{u})$$

$$\boxed{\delta f(\vec{x},\vec{u}) = \delta\vec{x}(t) \approx A\vec{x}(t) + B\vec{u}(t)}$$

$$\nabla_x f \Big|_{x^*, u^*} \qquad \nabla_u f \Big|_{x^*, u^*}$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\vec{x} = \vec{x}^*} \begin{bmatrix} x_1 - x_1^* \\ \vdots \\ x_n - x_n^* \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{\vec{x} = x^*} (x_1 - x_1^*) + \frac{\partial f_1}{\partial x_2}\Big|_{\vec{x}^*} (x_2 - x_2^*) + \cdots + \frac{\partial f_1}{\partial x_n}\Big|_{\vec{x}^*} \delta x_n \\ \vdots \\ \frac{\partial f_n}{\partial x_1}\Big|_{\vec{x} = x^*} (x_1 - x_1^*) + \cdots + \frac{\partial f_n}{\partial x_n}\Big|_{\vec{x} = x^*} (x_n - x_n^*) \end{bmatrix}$$

Dis 5A Worksheet

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1 Single-dimensional linearization

This is an exercise in linearizing a scalar system. The scalar nonlinear differential equation we have is

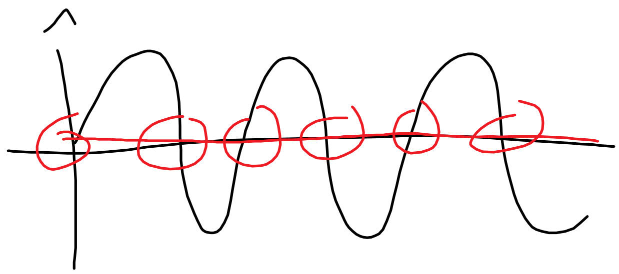
$$\frac{d}{dt}x(t) = \sin(x(t)) + u(t). \tag{1}$$

$$\dot{\vec{x}}(t) = f(\vec{x}^*, \vec{u}^*) = 0$$

$$\approx \underbrace{\cancel{f(\vec{x}^*, \vec{u}^*)}}_{0} + A\vec{x} + B\vec{u}$$

a) Find the equilibrium points for $u^* = 0$. You can do this by sketching $\sin(x)$ for $-4\pi \le x \le 4\pi$ and intersecting it with the horizontal line at 0. This will give you the equilibrium points x^* where $sin(x^*) + u^* = 0$.

$$\vec{x}^* = m\pi, \text{ for } m \in \mathbb{Z}$$



$$u^* = 0: \quad f(x, u) = \sin x + u$$

$$f(x^*, u^*) = 0 = \sin(x^*) + \cancel{u^*}$$

$$= 0 = \sin(x^*)$$

b) Linearize the system (1) around the equilibrium $(x_0^*, u^*) = (0, 0)$. **What is the resulting linearized scalar differential equation for $\tilde{x}(t) = x(t) - x_0^* = x(t) - 0$, involving $\tilde{u}(t) = u(t) - u^* = u(t) - 0$?**

$$\tilde{x}(t) = x(t) - x_0^* = \delta x$$

$$\tilde{u}(t) = u(t) - u^* = \delta u$$

$$f(x, u) = \sin(x) + u$$

$$f(x, u) = \underbrace{\cancel{f(x_0^*, u^*)}}_{0} + \frac{\partial f}{\partial x}\Big|_{x_0^*, u^*} \underbrace{(x - x_0^*)}_{\delta x}$$

$$x_0^*, u^* \quad \partial x$$

0

$$+ \frac{\partial f}{\partial u}\bigg|_{x_0^*, u^*} \underbrace{(u - u^*)}_{\delta u}$$

$$f(x,u) \approx \frac{\partial}{\partial x}(\sin x + u)\bigg|_{x^*, u^*} \delta x + \frac{\partial}{\partial u}(\sin x + u)\bigg|_{x^*, u^*} \delta u$$

$$= \cos(x)\bigg|_{x^*, u^* = (0,0)} \delta x + 1 \times \delta u$$

$$\boxed{f(x,u) \simeq \delta x + \delta u = \tilde{x}(t) + \tilde{u}(t)}$$

2 Jacobian Warm-Up

Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

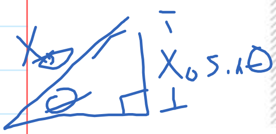
$$f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_1 x_2^2 \\ x_1 \end{bmatrix}$$

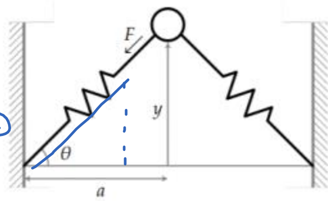
Calculate its Jacobian.

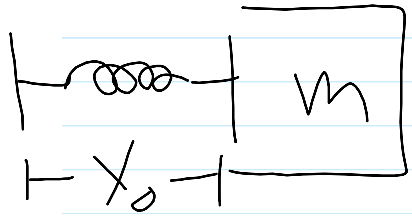
$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_1 + x_2^2 & 2x_1 x_2 \\ 1 & 0 \end{bmatrix}$$

3 Linearization

Consider a mass attached to two springs:







We assume that each spring is linear with spring constant k and resting length X_0. We want to build a state space model that describes how the displacement y of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt^2} = -\frac{2k}{m}\left(y - X_0\frac{y}{\sqrt{y^2+a^2}}\right)$. $= -\frac{2k}{m}\left(y - X_0 \sin\theta\right)$

a) Write this model in state space form $\dot{x} = f(x)$.

State variables: $x_1 = y$

$x_2 = \frac{dy}{dt}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{2k}{m}\left(y - X_0\frac{y}{\sqrt{y^2+a^2}}\right) \end{bmatrix}$$

$$\boxed{\dot{\vec{x}} = \begin{bmatrix} x_2 \\ -\frac{2k}{m}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right) \end{bmatrix}}$$

b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

Expect: $\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2^* \\ -\frac{2k}{m}\left(x_1^* - X_0\frac{x_1^*}{\sqrt{x_1^{*2}+a^2}}\right) \end{bmatrix}$$

$$\begin{bmatrix} \\ 0 \end{bmatrix} \quad \begin{bmatrix} \\ \frac{}{m}\left(x_1^* - x_0 \frac{x_1^*}{\sqrt{x_1^{*2}+a^2}}\right) \end{bmatrix}$$

$$\Rightarrow \quad x_2^* = 0$$

$$\Rightarrow \quad 0 = x_1^* - x_0 \frac{x_1^*}{\sqrt{x_1^{*2}+a^2}}$$

$$x_1^* = x_0 \frac{x_1^*}{\sqrt{x_1^{*2}+a^2}}$$

$$\Rightarrow \quad x_1^*\left(1 - \frac{x_0}{\sqrt{x_1^{*2}+a^2}}\right) = 0$$

$$\underline{\underline{x_1^* = 0}} \quad \text{or} \quad \frac{x_0}{\sqrt{x^{*2}+a^2}} = 1$$

$$\frac{x_0^2}{x_1^{*2}+a^2} = 1$$

$$\boxed{\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\Rightarrow x_1^{*2} = x_0^2 - a^2 < 0$$

But $x_0 < a$

~~x_1^* would be imaginary?~~

~~Doesn't make sense~~

~~Doesn't make sense~~

c) Linearize your model about the equilibrium.

$$\nabla f \in \mathbb{R}^{2\times 2}$$

$$\underbrace{\frac{d\vec{x}}{dt}}_{n\times 1} = A \underbrace{\vec{x}}_{n\times 1} \implies A \text{ has to be square}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$x_1^*, x_2^* = (0,0)$$

$$f_1(x_1, x_2) = x_2$$

$$f_2(x_1, x_2) = -\frac{2k}{m}\left(x_1 - X_0 \frac{x_1}{\sqrt{x_1^2 + a^2}}\right)$$

$$\nabla f = \begin{bmatrix} 0 & 1 \\ \left.\frac{\partial f_2}{\partial x_1}\right|_{0,0} & 0 \end{bmatrix}$$

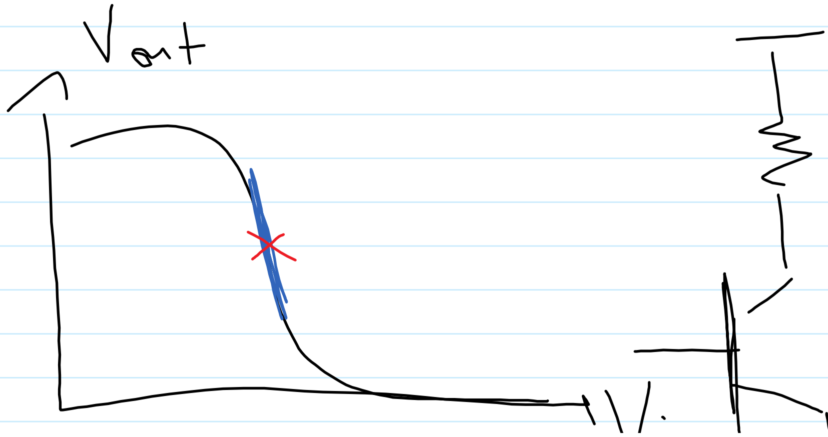
$$\left.\frac{\partial f_2}{\partial x_1}\right|_{0,0} = -\frac{2k}{m}\frac{\partial}{\partial x_1}\left(x_1 - X_0\frac{x_1}{\sqrt{x_1^2+a^2}}\right)$$

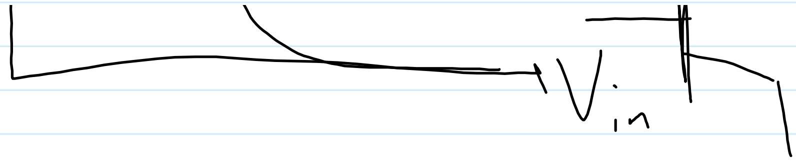
$$= -\frac{2k}{m}\left(1 - X_0\,\frac{1\cdot\sqrt{x_1^2+a^2} - \underbrace{x_1}_{=0}\cancel{\frac{\partial}{\partial x_1}\sqrt{x_1^2+a^2}}}{\left(\sqrt{x_1^2+a^2}\right)^2}\right)\Bigg|_{x_1,x_2=(0,0)}$$

$$= -\frac{2k}{m}\left(1 - X_0\frac{\sqrt{a^2}}{a^2}\right)$$

$$= -\frac{2k}{m}\left(1-\frac{X_0}{a}\right)$$

$$\Rightarrow \boxed{\vec{x}(t) \approx \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1-\frac{X_0}{a}\right) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

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