Dis 5B Notes

Monday, July 20, 2020 12:35 PM

Discretization + Discrete Time Systems

* Discrete Time Systems
* Discretization

(I) Discrete Time Systems

CT: $$\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), \vec{u}(t))$$

DT: $$\vec{x}[t+1] = f(\vec{x}[t], \vec{u}[t])$$

If linear,

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

$n \times 1 \quad n \times n \quad n \times 1 \quad n \times m \quad m \times 1$

$\vec{x}[t] \in \mathcal{X}$ ⟵ state space

$\vec{u}[t] \in \mathcal{U}$ ⟵ control space

* Equilibrium Points

CT: $\mathcal{X}_{eq}(\vec{u}) = \{\vec{x} \mid f(\vec{x}, \vec{u}) = 0\}$

Why? $\frac{d\vec{x}}{dt} = 0 \implies$ constant solutions

DT: $\boxed{\mathcal{X}_{eq} = \{\vec{x} \mid f(\vec{x}, \vec{u}) = \vec{x}\}}$

$\vec{x}[t+1] = f(\vec{x}, \vec{u}) = \vec{x}[t]$

$\hookrightarrow$ constant in time!

— If linear, solve for $\vec{x}^*$ in this eqn:

$$\vec{x}^* = A\vec{x}^* + B\vec{u}$$

$$I\vec{x}^* - A\vec{x}^* = B\vec{u}$$

$$\Rightarrow \underline{(I-A)\vec{x}^* = B\vec{u}}$$

Cases: unique sln, no sln, infinite slns

(1) $I-A$ is invertible

$$\Rightarrow \underbrace{\vec{x}^* = (I-A)^{-1}B\vec{u}}_{\text{unique s/n!}}$$

(2) $I-A$ is not invertible!

a) $B\bar{u} \in Col(I-A)$

$\implies$ infinite slns

$(\vec{x}_p + \vec{x}_h)$

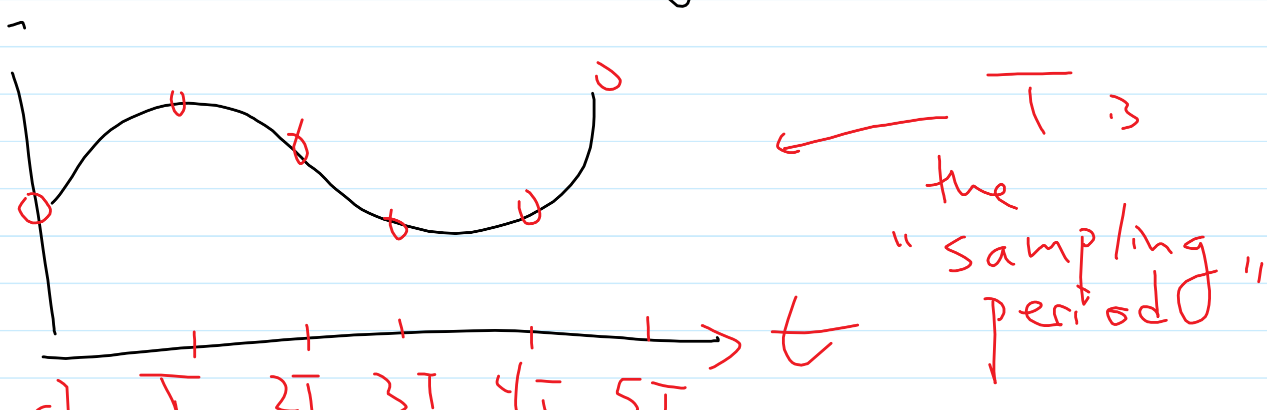
b) $B\bar{u} \notin Col(I-A)$

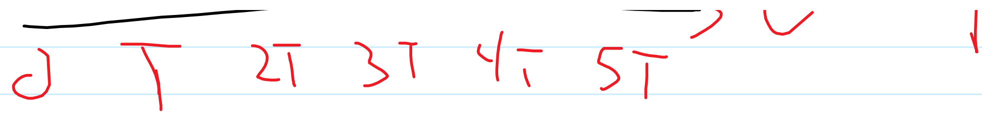
$\implies$ no slns

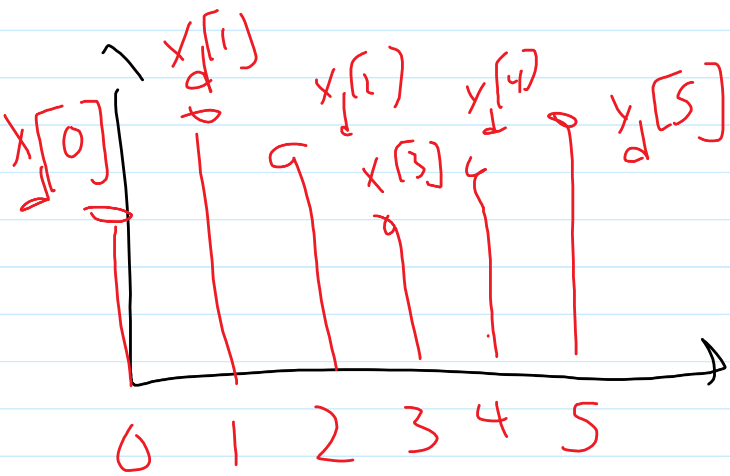
II. Discretization

How do you go from CT $\to$ DT?

Perhaps we sample the evolution of some CT system







"samples" or "timesteps"

$$x_d[n] = x(nT)$$ ⟵ sampled x at time $t = nT$

We will look at the case of a scalar diff eq w/ piecewise constant inputs?

Why? → digital control

$$\frac{dx}{dt} = \lambda x(t) + \mu u(t)$$

CT differential eqn

eqn

$$x_d[t+1] = a x_d[t] + b u_d[t]$$

DT difference eqn

Dis 5B Worksheet

Tuesday, July 21, 2020 11:23 AM

2 Dirty Dishes

I am a trip planner who lodges travellers at Bob's Bed and Breakfast. At the beginning of each day, Bob will do half of the dirty dishes in the sink. During the day, each of his guests will use 4 pounds of dishes minus an eighth pound of dishes for each pound of dishes already in the sink at the beginning of the day (as Bob's kitchen gets too messy).

a) What is the state vector for Bob's kitchen sink system? What are the inputs? Write out the state space model.

State variables: should characterize our system fully

$x[t]$ — pounds of dirty dishes

$u[t]$ — number of guests

$$\boxed{x[t+1] = \frac{1}{2}x[t] + \left(4 - \frac{1}{8}x[t]\right)u[t]}$$

b) Explain why Bob's kitchen is not a linear system.

Linear systems: $x[t+1] = \lambda x[t] + \mu u[t]$

$= a x[t] + b u[t]$

Pattern match: $a = \frac{1}{2}$

$b = 4 - \frac{1}{8}x[t]$

" $\frown$

c) On Wednesday morning (before Bob gets up), there are 4 pounds of dishes in the sink. On Wednesday, Bob has 4 guests, and on Thursday, he has 5 guests. How many pounds of dishes are in the sink after Thursday?

$x[t] = 4 \leftarrow$ Wed

$x[t] = 4 \leftarrow$ Wed

$x[t+1] = ? \leftarrow$ Thurs

$$x[t+1] = \frac{1}{2}x[t] + \left(4 - \frac{1}{8}x(t)\right)u[t]$$

$u[t] = 4$ guests

$$\Longrightarrow x[t+1] = \frac{1}{2}4 + \left(4 - \frac{1}{8}4\right)4$$

$$= 2 + \frac{7}{2}4 = 16$$

$x[t+2] = ?$

$u[t+1] = 5$

$$\Longrightarrow x[t+2] = \frac{1}{2}16 + \left(4 - \frac{1}{8}16\right)5$$

$$= 8 + 2 \cdot 5$$

$\boxed{x[t+2] = 18 \text{ lbs of d } \text{bles}}$

d) I am a very eccentric trip planner and I want Bob to have exactly 12 pounds of dishes in his sink. He has 24 pounds of dishes in his sink. How many guests should I lodge at Bob's Bed and Breakfast today? How many guests should I lodge tomorrow?

"planning"

What inputs are needed to keep the state at $X = 12$?

$$x[t] = 24$$

$$\longrightarrow x[t+1] = 12$$

$$x[t+2] = 12$$

$$x[t+1] = 12 = \frac{1}{2}x[t] + \left(4 - \frac{1}{8}x(t)\right)u[t]$$

$$= \frac{1}{2}24 + \left(4 - \frac{1}{8}24\right)u[t]$$

$$= 12 + u[t]$$

$$\Longrightarrow \boxed{u[t] = 0}$$

$$x[t+2] = 12 = \frac{1}{2}x[t+1] + \left(4 - \frac{1}{8}x(t+1)\right)u[t+1]$$

$$= 6 + \left(4 - \frac{12}{8}\right)u[t+1]$$

$$= 6 + \frac{5}{2}u[t+1]$$

$$6 = \frac{5}{2}u[t+1] \Longrightarrow \boxed{u[t+1] = \frac{12}{5}}$$

e) Now suppose 5 guests come to Bob's kitchen every day. At the equilibrium state, how many pounds of dishes will remain in the sink?

$\longrightarrow$ state stays the same

many pounds of dishes will remain in the sink?

$$(I - A)x^* = Bu$$

state stays the same

↳ scalar case:

$$x[t+1] = \frac{1}{2}x[t] + \left(4 - \frac{1}{8}x[t]\right)u[t]$$

Set $x[t] = x[t+1] = x^*$

$$\Longrightarrow x^* = \frac{1}{2}x^* + \left(4 - \frac{1}{8}x^*\right)u[t]$$

$$\frac{1}{2}x^* = \left(4 - \frac{1}{8}x^*\right)u[t]$$

But $u[t] = 5$

$$\frac{1}{2}x^* = 20 - \frac{5}{8}x^*$$

$$\frac{9}{8}x^* = 20 \Longrightarrow \boxed{x^* = \frac{160}{9}\text{ lbs}}$$

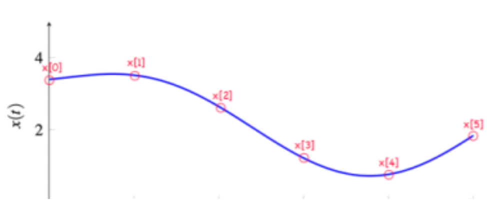
3 Differential equations with piecewise constant inputs

Let $x(\cdot)$ be a solution to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda\left(x(t) - u(t)\right). \tag{4}$$

Let $T > 0$. Let $x[\cdot]$ "sample" $x(\cdot)$ as follows:

$$x[n] = x(nT). \tag{5}$$



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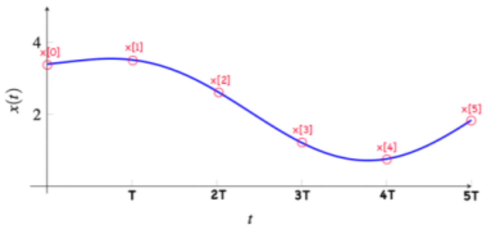
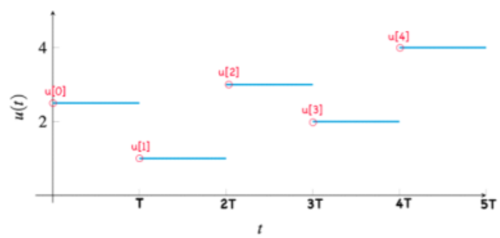
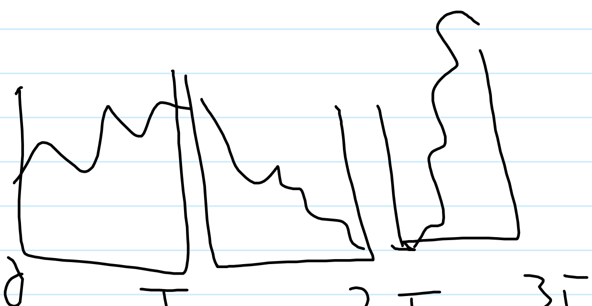


Assume that $u(\cdot)$ is constant between samples of $x(\cdot)$, i.e.

$$u(t) = u[n] \quad \text{when} \quad nT \le t < (n+1)T. \tag{6}$$



a) We will approach solving this differential equation iteratively in intervals of size T. What is the solution $x(t)$ for $t \in [0, T)$?



Why? then $u(t) = u[0]$ for $t \in [0, T)$

constant

Solve

$$\frac{dx}{dt}(t) = \lambda\left(x(t) - u(t)\right) \text{ for } t \in [0, T)$$

$$= \lambda\left(x(t) - u[0]\right)$$

"b"

$$\tilde{x}(t) = x(t) - u[0]$$

$$\frac{d\tilde{x}}{dt} = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{dx}{dt}$$

$$\Longrightarrow \frac{d\tilde{x}}{dt} = \lambda \tilde{x}$$

$$\Longrightarrow \tilde{x}(t) = \tilde{x}(0) e^{\lambda t}$$

$$\tilde{x}(t) = x(t) - u[0]$$

$$\tilde{x}(0) = x(0) - u[0]$$

$$\Longrightarrow \tilde{x}(t) = \left(x(0) - u[0]\right) e^{\lambda t}$$

$$\tilde{x}(t) = x(t) - u[0] = \left(x(0) - u[0]\right) e^{\lambda t}$$

$$\boxed{x[t] = e^{\lambda t} x(0) + \left(1 - e^{\lambda t}\right) u[0]}$$

b) Using your solution from the previous part, sample it at $t = T$ to write $x[1]$ in terms of $x[0]$ and $u[0]$.

Plug in $t = T$

Recall that $x[n] = x(nT)$

$$\Rightarrow x[1] = x(T)$$

$$x[1] = x(T) = x(0)e^{\lambda T} + (1 - e^{\lambda T})u[0]$$

$$x[1] = e^{\lambda T}x[0] + (1 - e^{\lambda T})u[0]$$

$$x[t+1] = ax[n] + bu[n]$$

c) Now for a general time-step n, write $x[n+1]$ in terms of $x[n]$ and $u[n]$. Conclude that the sampled system of a continuous-time linear system is in fact a discrete-time linear system.

$$t \in (nT, (n+1)T)$$

$$x(nT) = x[n]$$

$$x((n+1)T) = x[n+1]$$

Same trick:

$$\frac{dx(t)}{dt} = \lambda(x(t) - u[n]) \text{ for } t \in (nT, $$

$$\frac{dx(t)}{dt} = \lambda\big(x(t) - u[n]\big) \quad \text{for } t \in (nT, (n+1)T)$$

$$\tilde{x} = x - u[n]$$

$$\tilde{x}(nT) = x(nT) - u[n]$$

$$\frac{d\tilde{x}}{dt} = \lambda \tilde{x}$$

$$\Longrightarrow \quad \tilde{x}(t) = \tilde{x}(nT)\, e^{\lambda(t-nT)}$$

$$x(t) - u[n] = \big(x(nT) - u[n]\big) e^{\lambda(t-nT)}$$

$$x(t) = e^{\lambda(t-nT)} x(nT) + \left(1 - e^{\lambda(t-nT)}\right) u[n]$$

Eval at $t = (n+1)T$

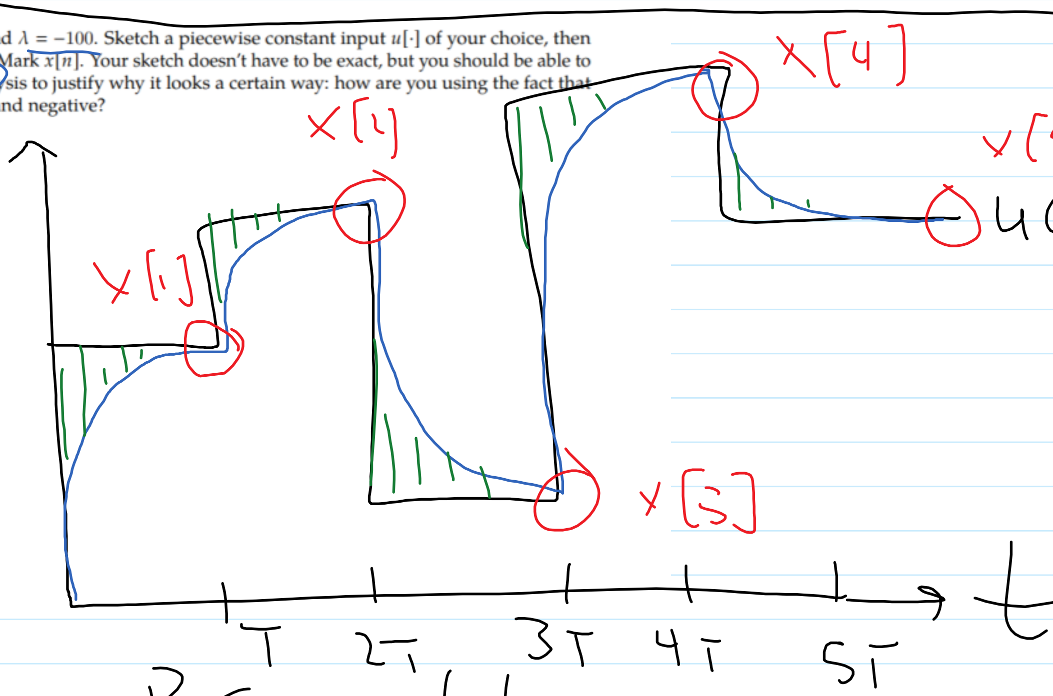
$$x((n+1)T) = e^{\lambda T} x(nT) + \left(1 - e^{\lambda T}\right) u[n]$$

$$\boxed{\begin{aligned} x[n+1] &= e^{\lambda T} x[n] + \left(1 - e^{\lambda T}\right) u[n] \\ a &= e^{\lambda T} \\ b &= \left(1 - e^{\lambda T}\right) \end{aligned}}$$

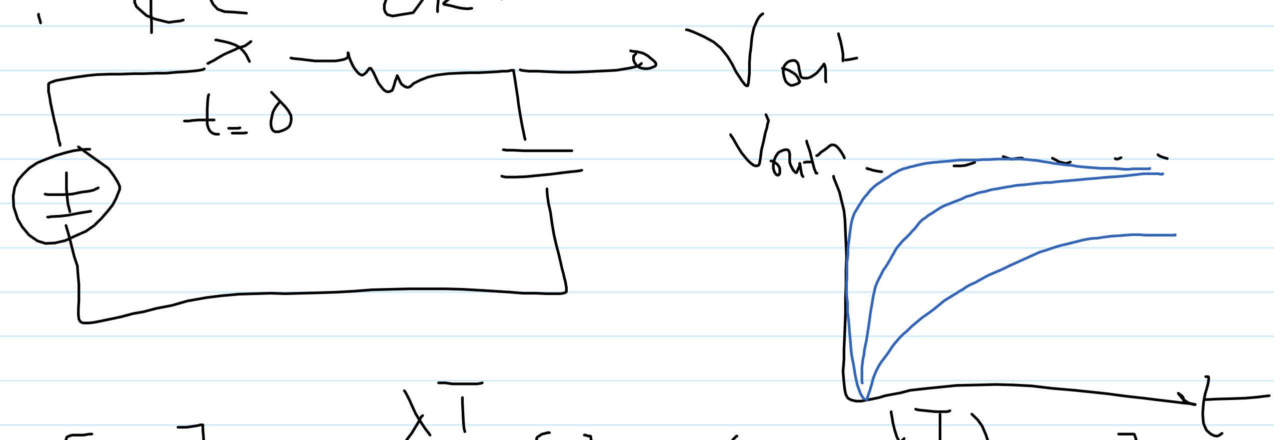
d) Let $T = 1$ and $\lambda = -100$. Sketch a piecewise constant input $u[\cdot]$ of your choice, then

$\to x[n]$

d) Let $T = 1$ and $\lambda = -100$. Sketch a piecewise constant input $u[\cdot]$ of your choice, then sketch $x(t)$. Mark $x[n]$. Your sketch doesn't have to be exact, but you should be able to supply analysis to justify why it looks a certain way: how are you using the fact that λT is large and negative?



Ex: RC ckt



(1) $x[n+1] = e^{\lambda T} x[n] + (1 - e^{\lambda T}) u[n]$

$\lambda T = -100 \times 1 = -100$

$= e^{-100} x[n] + (1 - e^{-100}) u[n]$

$x[n+1] \approx u[n]$

(2) $\tilde{x}(t) = x(t) - u[n]$

$\Rightarrow \tilde{x}(t) \approx \tilde{x}(t_0)\, e^{\lambda t}$

"displacement from the input".

always exp. decays

(3) if RC ckt,

$e^{\lambda T} = e^{-t/\tau}$

$\Longrightarrow \tau = \frac{1}{100}\ s$

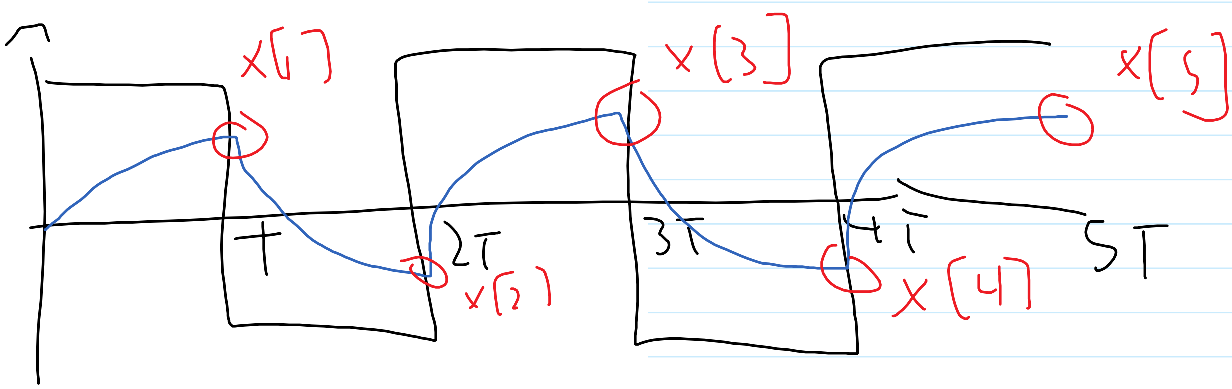
if $T = 1s$, $T \gg \tau$

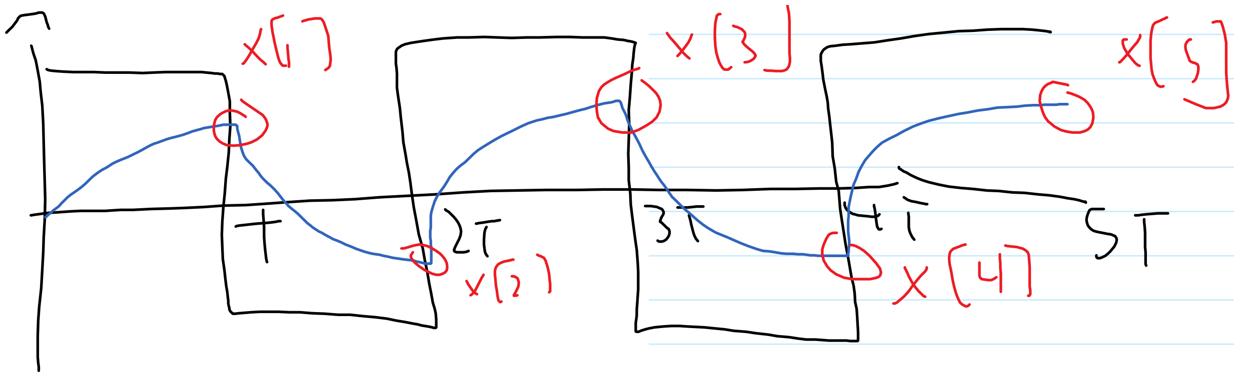
e) Let $T = 1$ and $\lambda = -1$. Define $u[n]$ as follows:

$$u[n] = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases} . \tag{7}$$

Sketch $x(t)$.

$\lambda T = -1$




x[1]
x[2]
x[3]
x[4]
x[5]
T
2T
3T
4T
5T