Dis 5C Notes

Wednesday, July 22, 2020 12:49 PM

* Discretization

* Controllability

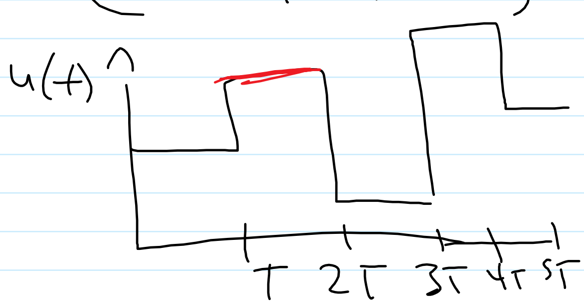
I. Discretization

Boils down to solving differential eqns with piecewise constant inputs

$\Rightarrow$ General technique: solve these DEs on a specific time interval $(nT, (n+1)T)$ where $u(t) = u[n]$ $= \text{const.}$ (on that interval)

$x[n] = x(nT)$

"timesteps" or "samples" — time



a) Scalar case

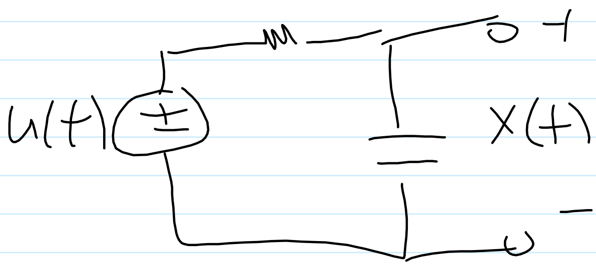
If $\frac{dx(t)}{dt} = \lambda x(t) + u\, u(t)$

From lecture / yesterday's discussion.

$$\boxed{x[n+1] = e^{\lambda T} x[n] + \mu \left(\int_0^T e^{\lambda s}\, ds \right) u[n]}$$

$$\int_0^T e^{\lambda s}\, ds = \begin{cases} \lambda = 0, & T \\ \lambda \neq 0, & \dfrac{e^{\lambda T} - 1}{\lambda} \end{cases}$$

Ex: RC ckt



$$\dot{v}_c(t) = -\frac{1}{RC} v_c(t) + \frac{1}{RC} u(t)$$

$$\lambda = -\frac{1}{RC},$$

$$\mu = \frac{1}{RC}$$

$$x[n+1] = e^{-T/RC} x[n] + \frac{1}{RC} \left(\int_0^T e^{\lambda s}\, ds \right) u[n]$$

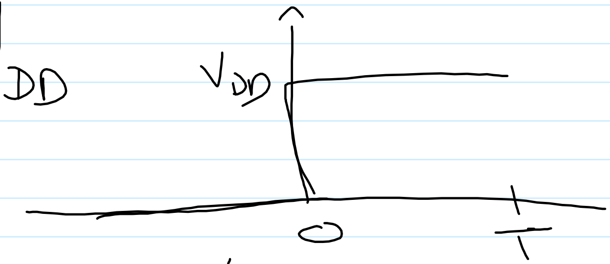
$$= e^{-T/RC} x[n] + \frac{1}{\cancel{R}C} \, \frac{e^{-T/RC} - 1}{-\frac{1}{\cancel{RC}}} u[n]$$

$$\underbrace{x[n+1] = e^{-T/RC} x[n] + \left(1 - e^{-T/RC}\right) u[n]}$$

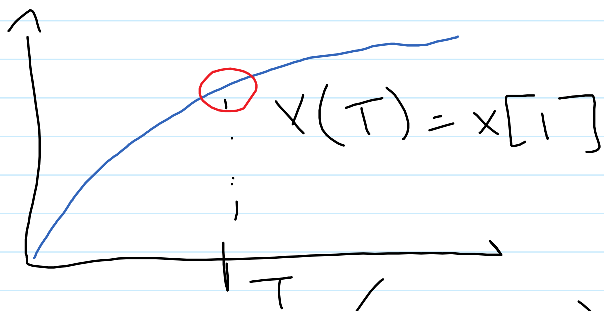
$$x[0] = x(0T) = v_c(0T) = 0$$

Supposing C is initially uncharged

$$u[0] = V_{DD}$$



$$x[n+1] = V_{DD}\left(1 - e^{-T/RC}\right) = v_c(T)$$



b) Vector case (linear)

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + B\vec{u}(t)$$

$\boxed{i}$ A is diagonalizable,

$$A = V\Lambda V^{-1}$$

$\Longrightarrow$ As before, "decouple" the sys of DEs using diagonalization

$$\frac{d\vec{x}}{dt} = A\vec{x} + B\vec{u} = V\Lambda V^{-1}\vec{x} + B\vec{u}$$

$$V^{-1}\frac{d\vec{x}}{dt} = \Lambda V^{-1}\vec{x} + V^{-1}B\vec{u}$$

Define $\vec{z} = V^{-1}\vec{x}$

$$\Longrightarrow \frac{d\vec{z}}{dt} = \Lambda\vec{z} + V^{-1}B\vec{u}$$

For simplicity, assume $B = \vec{b}$ ($n \times 1$ vector)

$\vec{u} = u$ (scalar)

$$\frac{d\vec{z}}{dt} = \Lambda\vec{z} + \vec{\tilde{b}}u \qquad \tilde{b} = V^{-1}\vec{b}$$

$$\begin{bmatrix} \frac{dz_1}{dt} \\ \vdots \\ \frac{dz_n}{dt} \end{bmatrix} = \begin{bmatrix} \lambda_1 z_1 \\ \vdots \\ \lambda_n z_n \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 u(t) \\ \vdots \\ \tilde{b}_n u(t) \end{bmatrix}$$

$\Longrightarrow$ n scalar diff eqs (which we've already seen!)

$$z_1[n+1] = e^{\lambda_1 T} z_1[n] + \int_0^T e^{\lambda_1 s}\,ds\; \tilde{b}_1 u[n]$$

$$z_1[n+1] = e^{\lambda_1 T} z_1[n] + \int_0^T e^{\lambda_1 s} ds\, \tilde{b}_1 u[n]$$

$$\vdots$$

$$z_n[n+1] = e^{\lambda_n T} z_n[n] + \int_0^T e^{\lambda_n s} ds\, \tilde{b}_n u[n]$$

In matrix form,

$$\vec{z}[n+1] = \begin{bmatrix} e^{\lambda_1 T} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n T} \end{bmatrix} \vec{z}[n] + \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & 0 \\ & \ddots & \\ 0 & & \int_0^T e^{\lambda_n s} ds \end{bmatrix} \underbrace{V^{-1}B}_{\tilde{b}} u[n]$$

$$\vec{x} = V\vec{z} \quad \left(\vec{z} = V^{-1}\vec{x} \right)$$

$$\Rightarrow \boxed{\vec{x}[n+1] = \underbrace{V e^{\Lambda T} V^{-1}}_{A_d} \vec{x}[n] + \underbrace{V \Lambda_T V^{-1} B}_{B_d} u[n]}$$

$$e^{\Lambda T} = \begin{bmatrix} e^{\lambda_1 T} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n T} \end{bmatrix}$$

$$\Lambda_T = \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & 0 \\ & \ddots & \\ 0 & & \int_0^T e^{\lambda_n s} ds \end{bmatrix}$$

[ii] A ... diagonalizable?

iii A is not diagonalizable?

(Something about matrix exponentials)

$\Rightarrow$ But sometimes the <u>diff eq</u> can be solved <u>directly</u> using other methods. Today, we will do this by directly integrating

$\Rightarrow$ Q2 on worksheet

II. Controllability

Controllable = we can reach <u>any</u> possible output state after a finite number of time steps, given some set of inputs

$\{u[0] \ldots u[k]\}$

Common question on exams, HWs, discs:

"Is this system controllable?"

Check controllability matrix $\mathcal{C}$:

$$\boxed{\begin{array}{l} \operatorname{rank} \mathcal{C} = n \iff \text{controllability} \\ n = \dim \mathcal{X} \quad (\text{state space}) \end{array}}$$

Derivation:

$$\vec{x}[1] = A\vec{x}[0] + B\vec{u}[0]$$

$$\vec{x}[2] = A\vec{x}[1] + B\vec{u}[1]$$

$$= A^2\vec{x}[0] + AB\vec{u}[0] + B\vec{u}[1]$$

$$\vdots$$

$$\vec{x}[k] = A^k\vec{x}[0] + A^{k-1}B\vec{u}[0] + \dots + AB\vec{u}[k-2]$$

So, $\underbrace{\vec{x}[k] - A^k \vec{x}[0]}$ $= \begin{bmatrix} A^{k-1}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[k-2] \\ \vec{u}[k-1] \end{bmatrix}$ $+ B\vec{u}[k-1]$

By def'n, the possible values are the column space of

$$\begin{bmatrix} A^{k-1}B & \cdots & AB & B \end{bmatrix}$$

Controllability matrix: $k = n$, $n = \dim \mathcal{X}$

$$\boxed{\mathcal{C} = \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix}}$$

Q: Why are n steps enough?

A: Cayley-Hamilton Theorem

↳ A corollary of this theorem is

that A^n, where A is an $n \times n$ matrix, can be written as a lin. comb. of lower matrix powers of A

$$\Longrightarrow A^n = c_{n-1} A^{n-1} + \dots + c_1 A + c_0 I$$

Thus, adding higher powers of A to $\mathcal{C}$ (going past n inputs), won't change the rank of $\mathcal{C}$

Ex: $\vec{x}[n+1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[n]$

$$\mathcal{C} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$\text{rank}\, \mathcal{C} = 1 < n = 2$

Not controllable? Why?

Intuition: $x_1[n+1] = x_1[n]$

$x_2[n+1] = x_2[n] + u[n]$

x_1 and x_2 are independent of each other.

Therefore, while $u[n]$ can affect x_2 explicitly, $u[n]$ cannot change x_1, whether explicitly or implicitly.

Concept check: Does uncontrollable mean we can't make the state go anywhere?

A: Not necessarily: just means we can't go everywhere (can still travel to any $\bar{x} \in \text{span } \mathcal{E}$)

Dis 5C Worksheet

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2 Deadbeat Control

Consider the system

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t].$$

a) Is this system controllable?

$$\mathcal{C} = \begin{bmatrix} AB & B \end{bmatrix} = \left[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \Longrightarrow \text{full rank } (\text{rank } \mathcal{C} = 2)$$

$$\boxed{\text{Controllable!}}$$

indirectly affects x_1

$$x_1[t+1] = x_1[t] - x_2[t]$$

$$x_2[t+1] = -x_1[t] + x_2[t] + u[t]$$

b) For which initial states $\vec{x}[0]$ is there a control that will bring the state to zero in a single time step?

Find $\vec{x}(0)$ s.t. $\vec{x}[1] = \vec{0}$

$$\vec{x}[1] = A\vec{x}[0] + B\vec{u}[0]$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} + \underline{\begin{bmatrix} 0 \\ u[0] \end{bmatrix}}$$

$$0 = x_1[0] - x_2[0] \quad (1)$$

$$0 = -x_1[0] + x_2[0] + u[0] \quad (2)$$

From (1), $\begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

From (2), $u[0] = 0$

$$\boxed{\vec{x}[0] = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$ ← null space of A!

c) For which initial states $\vec{x}[0]$ is there a control that will bring the state to zero in two time steps?

$\vec{x}[2] = \vec{0}$

From def'n of controllability, for a state space of $\dim \mathcal{X} = n$, we can reach any possible state in n timesteps

$\Longrightarrow \quad \dim \mathcal{X} = 2$

So we can reach any output, including $\vec{x}[2] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in 2 timesteps, from any initial state

$$\Rightarrow \boxed{\vec{x}[0] = \text{anything}}$$

$$\vec{x}[2] - A^2 \vec{x}[0] = \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

$$\Downarrow$$

$$\vec{x}[2] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} + \begin{bmatrix} -u[0] \\ u[0]+u[1] \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_1[0] - 2x_2[0] - u[0] \\ -2x_1[0] + 2x_2[0] + u[0] + u[1] \end{bmatrix}$$

$u[0]$ affects both $x_1[2], x_2[2]$

$u[1]$ only affects $x_2[2]$

Pick whatever vals of $u[0], u[1]$ solve this!

d) Now let $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be the initial state. Give a set of control inputs $u[0]$ and $u[1]$ to bring to system to $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in two time steps.

$$\vec{x}[2] = A^2 \vec{x}[0] + \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

$$\vec{x}[2] = \vec{y}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

$$\boxed{u[0] = 2, \quad u[1] = 1}$$

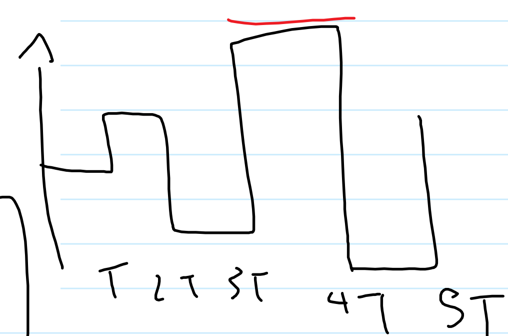
3 Discretization

Consider a cart of mass M, pushed with a force $u(t)$ with position, $x(t)$, and velocity, $v(t)$. Hence, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = v(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = \frac{u(t)}{M}$$

vector diff eq

We will apply a constant input between any time $t \in [nT, nT + T)$. Here T is our time between samples.



a) Find a discretized system of equations for this system.

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t)$$

not diagonalizable!

Are we done?

↳ Solve DEs directly!

$$\dot{x}(t) = v(t)$$

$$\dot{v}(t) = \frac{u(t)}{M} = \frac{u[n]}{M}$$

constant on interval $t \in (nT, (n+1)T)$

Solve this first!

$$\int_{v(nT)}^{v(t<(n+1)T)} dv = \int_{nT}^{t} \frac{u[n]}{M} \, dt'$$

$$\underbrace{v(t) - v(nT) = \frac{u[n]}{M}(t - nT)}$$

$$\frac{dx}{dt} = v(t) = v(nT) + \frac{u[n]}{M}(t - nT)$$

$$\int_{x(nT)}^{x(t)} dx = \int_{nT}^{t} v(nT) + \frac{u[n]}{M}(t' - nT) \, dt'$$

$$x(t) - x(nT) = v(nT)(t - nT)$$

$$+\frac{u[n]}{M}\left(\frac{t'^2}{2}\Big|_{nT}^{t} - nT\, t'\Big|_{nT}^{t}\right)$$

$$= v(nT)(t-nT)$$

$$+\frac{u[n]}{M}\left(\frac{t^2}{2}-\frac{(nT)^2}{2}-nTt+(nT)^2\right)$$

$$\underbrace{= v(nT)(t-nT)+\frac{u[n]}{2M}(t-nT)^2}$$

Eval $x(t), v(t)$ at $t = nT + T$

$$\Rightarrow x((n+1)T) = x(nT) + v(nT)T + \frac{u[n]}{2M}T^2$$

$$v((n+1)T) = v(nT) + \frac{u[n]}{M}T$$

Final step to convert CT $\rightarrow$ DT?

$$x[n] = x(nT)$$

$$x[n+1] = x[n] + Tv[n] + \frac{T^2}{2M}u[n]$$

$$x[n+1] = x[n] + T v[n] + \frac{T^2}{2M} u[n]$$

$$v[n+1] = v[n] + \frac{T}{M} u[n]$$

$$\boxed{\begin{bmatrix} x[n+1] \\ v[n+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A_d} \begin{bmatrix} x[n] \\ v[n] \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{T^2}{2M} \\ \frac{T}{M} \end{bmatrix}}_{B_d} u[n]}$$

b) Is the discretized system controllable?

$$\mathcal{C} = \begin{bmatrix} AB & B \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}\frac{T^2}{M} & \frac{T^2}{2M} \\ \frac{T}{M} & \frac{T}{M} \end{bmatrix}$$

Full rank? $\boxed{\text{yes}}$

Arg1: if not full rank, det $= 0$

$$\Rightarrow \det = \frac{3}{2}\frac{T^2}{M}\frac{T}{M} - \frac{T^2}{2M}\frac{T}{M}$$

$$= \frac{T^3}{M^2} \neq 0 \quad \text{unless } T=0$$

(which is nonsensical)

Arg 2: cols are indep. unless

$$\frac{3}{2}\frac{T^2}{M} = \frac{1}{2}\frac{T^2}{M}$$

$\Rightarrow$ only true if $T=0$

(still nonsensical)