Discussion 5D Notes

Minimum Energy Control

- SVD and Compact SVD
- Least Squares
- Minimum Energy Control

I.) SVD

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T} \qquad (A \in \mathbb{R}^{m\times n})$$

rank $A = r$

eigenvectors

AA^T: $\underbrace{\vec{u}_1 \cdots \vec{u}_r}_{\text{span Col}(A)}, \underbrace{\vec{u}_{r+1} \cdots \vec{u}_m}_{\text{span Null}(A^T)}$ } span $\mathbb{R}^m$

A^TA: $\underbrace{\vec{v}_1 \cdots \vec{v}_r}_{\text{span Col}(A^T)}, \underbrace{\vec{v}_{r+1} \cdots \vec{v}_n}_{\text{span Null}(A)}$ } span $\mathbb{R}^n$

Remember: these 1st r singular vectors are those that correspond to

are those that correspond to
the r nonvanishing singular values
$(\sigma_i \neq 0)$

Let us write the <u>compact SVD</u>:

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

Turns out we can write this in matrix form:

$$A = \underbrace{U_c}_{m\times r} \underbrace{\Sigma_c}_{r\times r} \underbrace{V_c^T}_{r\times n}$$

* U_c composed of the singular vectors spanning $Col(A)$: $\vec{u}_1 \dots \vec{u}_r$
* V_c composed of the singular vectors spanning $Col(A^T)$: $\vec{v}_1 \dots \vec{v}_r$
* Σ_c: diagonal entries are the nonvanishing singular values $\sigma_i \neq 0$

Often helpful to write the full SVD in
block matrix form to include the "compact"

Block matrix [illegible] to [illegible] compact matrices:

$$A = U\Sigma V^T = [U_c \; U_2] \begin{bmatrix} \Sigma_c & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} V_c^T \\ V_2^T \end{bmatrix}$$

Note if we do the multiplication:

$$A = [U_c \; U_2] \begin{bmatrix} \Sigma_c V_c^T + 0 \\ 0 + 0 \end{bmatrix}$$

$$= [U_c \; U_2] \begin{bmatrix} \Sigma_c V_c^T \\ 0 \end{bmatrix}$$

$$= U_c \Sigma_c V_c^T$$

* "Extreme" Cases

a) A is a tall matrix of rank n ($m > n$, lin. ind. columns)

$$A = [U_c \; U_2] \begin{bmatrix} \Sigma_c & 0_{r\times(n-r)} \end{bmatrix} \begin{bmatrix} V_c^T \\ V_2^T \end{bmatrix}$$

$$\begin{bmatrix} 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

But rank $A = r = n$ (b/c lin. ind. columns)

So $0_{n\times(n-r)} = 0_{n\times 0}$ empty!

$0_{(m-r)\times(n-r)} = 0_{(m-r)\times 0}$ empty!

Moreover, $V \in \mathbb{R}^{n\times n}$

$V_c \in \mathbb{R}^{n\times r} \Rightarrow \mathbb{R}^{n\times n}$

$(r = n)$

So $V_c = V$!

$$\therefore \boxed{A = [U_c \; U_2] \begin{bmatrix} \Sigma_c \\ 0 \end{bmatrix} V^T}$$

b) A is a wide matrix of rank m ($m < n$, lin. ind. rows)

$$\boxed{A = U [\Sigma_c \; 0] \begin{bmatrix} V_c^T \\ V_2^T \end{bmatrix}}$$

$(U = U_c)$

Why do we care about these extreme cases?

- tall matrix (overdetermined system)

$\Longrightarrow$ Least Squares

- wide matrix (underdetermined systems)

$\Longrightarrow$ min norm

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$ ⟵ l.n. ind. columns

$$\mathcal{U} = \begin{pmatrix} \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{11}} \\ \sqrt{\frac{2}{11}} & 0 & \frac{3}{\sqrt{11}} \end{pmatrix} = \left[\underbrace{\begin{pmatrix} \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{11}} & 0 \end{pmatrix}}_{\mathcal{U}_c} \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{pmatrix}}_{\mathcal{U}_2} \right]$$

$$\Sigma = \begin{pmatrix} \sqrt{11} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{11} & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{pmatrix} \quad \Sigma_c$$

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = V_c^T$$

$$A = U \Sigma V^T = U_c \Sigma_c V_c^T$$

$$= \begin{pmatrix} \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{11}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{11} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \sqrt{11} \begin{pmatrix} \frac{3}{\sqrt{22}} \\ \frac{3}{\sqrt{22}} \\ \sqrt{\frac{2}{11}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} + (1) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

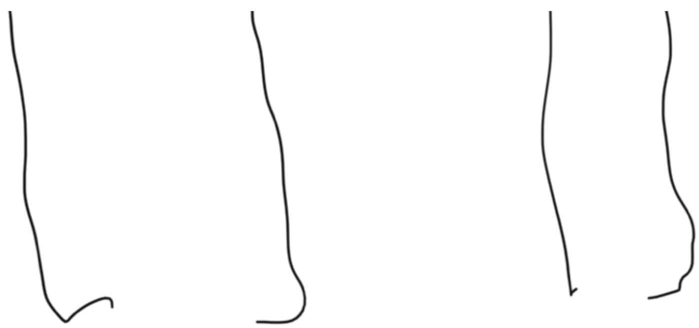
II. SVD and Least Squares

overdetermined systems, lin. ind. columns

$$A\vec{x} \approx \vec{y}$$

$$\begin{array}{cc} 1 & 2 \end{array}$$
$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

1000 linear eqns

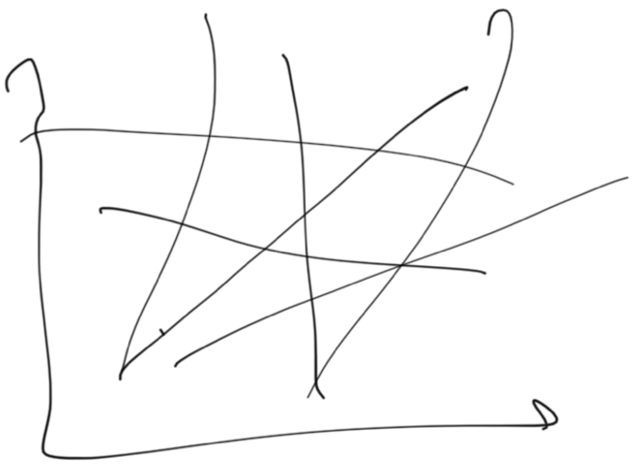
1000 

(too many constraints)

$\Downarrow$

no solution

(most of the time)



Back to $A\vec{x} \approx \vec{y}$ $\longleftarrow$ could happen for example if you take a bunch of noisy measurements, so no exact solution

Optimization Problem:

$$\vec{x}^* = \min_{x} \|\vec{e}\|^2 \Longrightarrow \underbrace{\vec{x}^* = \min_{\vec{x}} \|A\vec{x} - \vec{y}\|^2}$$

$$A\vec{x} + \vec{e} = \vec{y}$$

$$\boxed{\begin{aligned} \vec{x}^* &= V\Sigma_c^{-1}U_c^T\vec{y} \\ &= (A^TA)^{-1}A^T\vec{y} \end{aligned}}$$

from 16A

Why?

$$\|\vec{e}\| = \|A\vec{x} - \vec{y}\| = \|U\Sigma V^T\vec{x} - \vec{y}\|$$

Remember: U is unitary — length preserving

$$\langle \vec{x}, \vec{x} \rangle = \langle U\vec{x}, U\vec{x} \rangle = x^T U^T U x = x^T x = \langle x, x \rangle$$

Thus we multiply $A\vec{x} - \vec{y}$ by U

$$\|\vec{e}\| = \|U^T U \Sigma V^T \vec{x} - U^T \vec{y}\|$$

$$= \|\Sigma V^T \vec{x} - U^T \vec{y}\|$$

$$= \left\| \begin{bmatrix} \Sigma_c \\ 0 \end{bmatrix} V^T \vec{x} - \begin{bmatrix} U_c^T \\ U_2^T \end{bmatrix} \vec{y} \right\|$$

$$= \left\| \begin{matrix} \Sigma_c V^T \vec{x} & - U_c^T \vec{y} \\ 0 & - U_2^T \vec{y} \end{matrix} \right]$$

$$= \left\| \begin{matrix} \Sigma_c V^T \vec{x} - U_c^T \vec{y} \\ -U_2^T \vec{y} \end{matrix} \right\|$$

this is fixed: we can't change this w/ our choice of $\vec{x}$

We can minimize the effect of this w/ our choice of $\vec{x}$

$$\left\| \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \right\| = \sqrt{x_1^2 + x_2^2 + \dots + x_k^2}$$

To minimize the contribution of

the k^{th} element, best we can do is set $x_k = 0$

So we set $\Sigma_c V^T \vec{x}^* - U_c^T \vec{y} = 0$

$$\Rightarrow \Sigma_c V^T \vec{x}^* = U_c^T \vec{y}$$

invertible!

$$\underbrace{\vec{x}^* = V \Sigma_c^{-1} U_c^T \vec{y}}$$

III. Minimum Norm Solution and SVD

undetermined systems, lin. ind. rows

$A\vec{x} = \vec{y}$ ⟵ infinite solutions, which is the best?

$$\vec{x}^* = \min \|\vec{x}\| \quad \text{s.t.} \quad \vec{y} = A\vec{x}$$

$$A\vec{x} = U \Sigma V^T \vec{x} = U \begin{bmatrix} \Sigma_c & 0 \end{bmatrix} V^T \vec{x}$$

$$\|V^T \vec{x}\| = \|\vec{x}\|$$

In block matrix form,

$$\|V^T \vec{x}\| = \left\| \begin{bmatrix} V_c^T \vec{x} \\ V_2^T \vec{x} \end{bmatrix} \right\| = \|\vec{x}\|$$

To minimize $\|\vec{x}\|$, probably want to set one of $V_c^T\vec{x}$ or $V_2^T\vec{x}$ to $\vec{0}$.

Which one of these is fixed?

Note: $U\Sigma V^T\vec{x} = \underset{\text{inv}}{U}\,\underset{\text{inv}}{\Sigma_c} V_c^T\vec{x} = \vec{y}$ $\quad (U_c = U)$ since full row rank

$$V_c^T\vec{x} = \Sigma_c^{-1}U^T\vec{y}$$

For $\vec{x}$ to be a solution, it must satisfy this eqn $\longrightarrow$ fixed

$$V^T\vec{x}^* = \begin{bmatrix} \overbrace{\Sigma_c^{-1}U^T\vec{y}}^{=V_c^T\vec{x}} \\ V_2^T\vec{x} \end{bmatrix}$$

Thus we pick $\vec{x}^*$ s.t. $V^T\vec{x}^* = \begin{bmatrix} \Sigma_c^{-1}U^T\vec{y} \\ 0 \end{bmatrix}$

$$\vec{x}^* = V\begin{bmatrix} \Sigma_c^{-1}U^T\vec{y} \\ 0 \end{bmatrix} = \begin{bmatrix} V_c & V_2 \end{bmatrix}\begin{bmatrix} \Sigma_c^{-1}U^T\vec{y} \\ 0 \end{bmatrix}$$

$$= V_c\Sigma_c^{-1}U^T\vec{y}$$

$$\boxed{\begin{aligned}\vec{x}^* &= V_c \Sigma_c^{-1} U^T \vec{y} \\ &= A^T (AA^T)^{-1} \vec{y}\end{aligned}}$$

1 Minimum Energy Warmup

$\vec{y} = A\vec{x}$, $A \in \mathbb{R}^{m \times n}$ $(m < n)$

$\Rightarrow$ lin. ind. rows

a) $\text{rank } A = m$

Why? row rank $= \dim \text{Col}(A^T) = m$

row rank $=$ column rank $=$ rank

$\Rightarrow$ $\text{rank } A = m$

b) i) Why is Σ_c invertible?

$$\begin{bmatrix} \Sigma_c & 0_{m \times (n-m)} \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_m & \end{array}\right]$$

$\sigma_1 \dots \sigma_r > 0$, $\sigma_{r+1} = \dots = \sigma_m = 0$

rank $A = r = m$,

$\sigma_1 \dots \sigma_m$ are all nonzero

Therefore, $\Sigma_c = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m \end{bmatrix}$ is invertible!

$$\boxed{\begin{aligned} \Sigma_{c,ij} &= \begin{cases} \sigma_i & i=j \\ 0 & i \neq j \end{cases} \\ \Sigma^{-1}_{c,ij} &= \begin{cases} \frac{1}{\sigma_i} & i=j \\ 0 & i \neq j \end{cases} \end{aligned}}$$

Alt. argument

$\Sigma = \begin{bmatrix} \Sigma_c & 0 \\ 0 & 0 \end{bmatrix}$

But if $r = m$, then this becomes

$\begin{bmatrix} \Sigma_c & 0 \end{bmatrix}$

invertible

c) $A^T(AA^T)^{-1} = V_c \Sigma_c^{-1} U^T$

Plug in $A = U\Sigma V^T$

$$A^T(AA^T)^{-1} = (U\Sigma V^T)^T (U\Sigma V^T (U\Sigma V^T)^T)^{-1}$$

$$= V\Sigma^T U^T (U\Sigma \underline{V^T V} \Sigma^T U^T)^{-1}$$

$$= V\Sigma^T U^T (U\Sigma\Sigma^T U^T)^{-1}$$

$$= V\Sigma^T U^T \left(U \begin{bmatrix} \Sigma_c & 0 \end{bmatrix} \begin{bmatrix} \Sigma_c^T \\ 0 \end{bmatrix} U^T\right)^{-1}$$

$$= V\Sigma^T U^T (U\Sigma_c \Sigma_c^T U^T)^{-1}$$

Note that Σ_c is diagonal, so $\Sigma_c = \Sigma_c^T$

$$= V\Sigma^T U^T (U\Sigma_c^2 U^T)^{-1}$$

Hint: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ for A, B, C inv.

$$= V\Sigma^T U^T (U^T)^{-1} (\Sigma_c^2)^{-1} U^{-1}$$

$$= V\Sigma^T \cancel{U^T U} (\Sigma_c^2)^{-1} U^T$$

$$= [V_c \; V_2] \begin{bmatrix} \Sigma_c \\ 0 \end{bmatrix} (\Sigma_c^2)^{-1} U^T$$

$$= V_c \Sigma_c (\Sigma_c^2)^{-1} U^T$$

$$(\Sigma_c^2)_{ij} = \begin{cases} \sigma_i^2 & i = j \\ 0 & i \neq j \end{cases}$$

$$= \underline{\underline{V_c \Sigma_c^{-1} U^T}}$$ As desired

$$\Sigma_c (\Sigma_c^2)^{-1} = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{pmatrix} \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \sigma_1 & & \\ & \ddots & \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \end{pmatrix}$$

$$\begin{pmatrix} \ddots & \\ & \sigma_m \end{pmatrix} \begin{pmatrix} \ddots & \\ & \frac{1}{\sigma_m^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_m} \end{pmatrix}$$

$$= \Sigma_c^{-1}$$

② Minimum Energy Control

$$\vec{x}(k+1) = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A} \vec{x}(k) + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u(k)$$

$$\vec{x}(5) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x}(0) = \vec{0}$$

a) $$\vec{x}(5) = A^5 \vec{x}(0) + \begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix} \begin{bmatrix} u(4) \\ u(3) \\ u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

$$= \vec{0} + \underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} u(4) \\ \vdots \\ u(0) \end{bmatrix}$$

$$\vec{x}(5) = \mathcal{C} \vec{u}$$

$$\vec{u}^* = \mathcal{C}^T (\mathcal{C}\mathcal{C}^T)^{-1} \vec{x}(5)$$

$$u = C^T (C C^T)^{-1} x(0)$$

$$\vec{u}^* = \begin{pmatrix} -0.2 \\ -0.1 \\ 0 \\ 0.1 \\ 0.2 \end{pmatrix}$$

$$\boxed{E = \|\vec{u}^*\|^2 = 0.1}$$

b) Technically, only need 2 steps?

$$\vec{x}(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{u}(2) = \vec{u}(3) = \dots = 0$$

Stay at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} 0$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In fact, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvec of A with eigenvalue 1

Thus, once we reach $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, if we apply 0 inputs, we just stay there

c) $u(0), u(1)$?

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

$$\mathcal{C} = [B \quad AB]$$

$$\Rightarrow \begin{array}{l} u(0) = 1 \\ u(1) = -1 \end{array}$$

$$\boxed{E = u(0)^2 + u(1)^2 = 2}$$

20x higher

③ Uncontrollability

$$\vec{x}(t+1) = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} u(t)$$

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

a) Controllable?

$$\mathcal{C} = (B \quad AB \quad A^2B)$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

rank $\mathcal{C} = 2$
(not controllable!)

Intuition?

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

$$\Rightarrow x_1(t+1) = 2x_1(t)$$

$$x_2(t+1) = -3x_1(t) + x_3(t)$$

$$x_3(t+1) = x_2(t) + 2u(t)$$

just keep growing

b) reach $\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$?

$\vec{x}(0)$

$$\vec{x}(1) = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} u(t)$$

$$= \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2u(t) \end{pmatrix}$$

Just pick $\boxed{u(0) = -1}$!

c) 2 time steps?

$$\vec{x}(2) = A^2\vec{x}(0) + \begin{bmatrix} B & AB \end{bmatrix}\begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} 4 \end{bmatrix}$$

$$= \begin{bmatrix} \vdots \\ -6+2u(0) \\ -3+2u(1) \end{bmatrix}$$

$u(0), u(1)$ are free variables

$$\boxed{\vec{x}(2) = \begin{bmatrix} 4 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}}$$

d) $\vec{x}(T) = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ at some time T?

$$\vec{x}(1) = \begin{pmatrix} 2 \\ -3 \\ 2u(0) \end{pmatrix}$$

$$\vec{x}(2) = \begin{pmatrix} 4 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\vec{x}(3) = \begin{pmatrix} 8 \\ \vdots \\ \vdots \end{pmatrix}$$

In fact, $x_1(t+1) = 2x(t)$

$$\Rightarrow x_1(T) = 2^T x_1(0) = 2^T(1) = 2^T$$

Grows exponentially!