Discussion 6B

Stability

* Discrete Time
* Continuous Time
* Example

(0.) Definition of Stability

Stable: $x(t)$ remains bounded for <u>any</u> initial condition and <u>any</u> bounded input sequence

Unstable: if $\exists$ initial condition and bounded input sequence such that $|x(t)| \to \infty$ as $t \to \infty$ (blows up)

Bounded:

$$|u(t)| \leq B < \infty \qquad \forall t$$

(I.) Discrete Time

a) Scalar case

$$x[k+1] = \lambda x[k] + b u[k]$$

$$\downarrow$$

$$x[k] = \lambda^k x[0] + \sum_{i=0}^{k-1} \lambda^{k-1-i} b u[i]$$

initial condition $x[0]$

bounded input sequence $u[i]$

- $|\lambda| > 1$: unstable

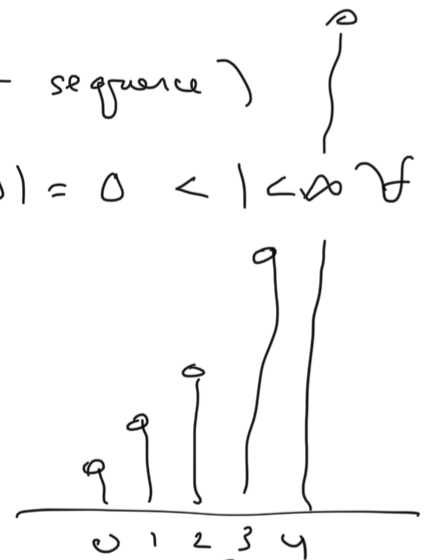
Pick $u = 0$ (bounded input sequence)

$\hookrightarrow$ $B = 1$, $|u(t)| = 0 < 1 < \infty \ \forall t$

Then $x[k] = \lambda^k x[0]$

If $x[0] \neq 0$

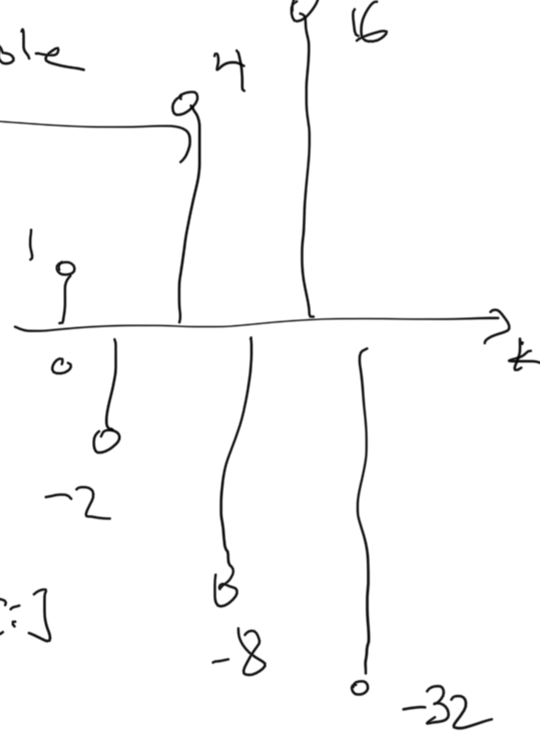
$\lim_{k \to \infty} |x[k]| \longrightarrow \infty$



Thus $|\lambda| > 1$ is unstable

$|\lambda| > 1$ but $\lambda < 0$

$\longrightarrow$ still blows up



Say $x[0] = 1$

$\lambda = -2$

- $|\lambda| < 1$: stable

$$x[k] = \underbrace{\lambda^k x[0]}_{S_1} + \underbrace{\sum_{i=0}^{k-1} \lambda^{k-1-i} b u[i]}_{S_2}$$

$\lim_{k \to \infty} S_1 = 0$

$\lim_{k \to \infty} S_2$?

But note that $|A| + |B| \geq |A+B|$

$$|S_2| = \left| \sum_{i=0}^{k-1} \lambda^{k-1-i} b u[i] \right| \leq \sum_{i=0}^{k-1} |\lambda|^{k-1-i} |b| |u[i]|$$

b is a constant, pull it out

$|u[i]| \leq B < \infty$

replaced $|u[i]|$ with its maximum value

$$\leq \underbrace{|b||B|}_{B'} \sum_{i=0}^{k-1} |\lambda|^{k-1-i}$$

$$= B' \sum_{i=0}^{k-1} |\lambda|^{k-1-i}$$

$j = k-1-i$

$$= B' \sum_{j=0}^{k-1} |\lambda|^{j}$$

Geometric series

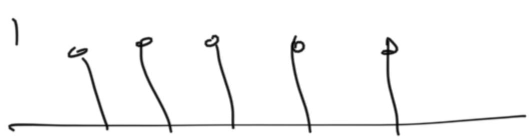
$$\lim_{k \to \infty} |S_2| \leq B' \sum_{j=0}^{\infty} |\lambda|^j = B' \frac{1}{1-|\lambda|}$$

Bounded! Thus, $|\lambda| < 1$ guarantees stability

- $|\lambda| = 1$

If there were no input, would be "stable"

$$|x[k]| = |\lambda|^k |x[0]| = |x[0]| \quad \text{if } |\lambda| = 1$$



$\lambda = 1$, $x[0] = 1$

But, if bounded input added, we can obtain an unbounded output

So in terms of "bounded input – bounded output"

So in terms of bounded input — bounded output stability (BIBO stability), $|\lambda| = 1 \longrightarrow$ unstable

For example,

$$x[k] = x[0] + \sum_{i=0}^{k-1} b\, u[i] \qquad (\lambda = 1)$$

Pick $u[i] = 1$

$$= x[0] + \sum_{i=0}^{k-1} b$$

$$\lim_{k \to \infty} x[k] \to \infty$$

So we say $|\lambda| = 1 \longrightarrow$ marginally stable

(Generally avoid this)

This generalizes to the vector case aswell;

- diagonalizable — just decouple into a system of independent scalar cases

$$\vec{x}[k+1] = A\,\vec{x}[k] + B\,u[k]$$

$$\vec{x}[k+1] = V \Lambda V^{-1}\,\vec{x}[k] + B\,u[k]$$

Define $z = V^{-1} x$ $\qquad \tilde{B} = V^{-1} B$

$$\vec{z}[k+1] = \Lambda\,\vec{z}[k] + \tilde{B}\,u[k]$$

$$\Rightarrow \quad z_i[k+1] = \lambda_i z_i[k] + \tilde{b}_i\,u[k] \qquad \forall\, i$$

Since scalar case from above, same conditions hold for vector hold for stability.

- Not diagonalizable

Can still define a transformation such that

$$\tilde{A} = T^{-1} A T \quad \text{is upper triangular}$$

$$\tilde{z}[k+1] = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ & \ddots & & \vdots \\ & & & * \\ 0 & & & \lambda_n \end{bmatrix} \tilde{z}[k] + \tilde{B} u[k]$$

Rough sketch:

instability: Note that you can pick the transformation T to have the λ_i be in any order on the diagonal.

$\Longrightarrow$ put $|\lambda_i| > 0$ at $i = n$

Thus, obtain $z_n(k+1) = \lambda_n z_n(k) + \tilde{b}_n u(k)$

Scalar case $\longrightarrow$ unstable

$\longrightarrow$ whole system unstable

stability: "go up the rows"

$|\lambda_n| < 1 \Longrightarrow z_n$ is bounded (from scalar case)

$n-1$ row: $z_{n-1}[k+1] = \lambda_{n-1} z_{n-1}[k] + \underbrace{\alpha z_n[k] + b_{n-1} u[k]}$

From scalar case, z_n is bounded, so

$\alpha z_n[k] + b_{n-1} u[k]$ is also bounded!

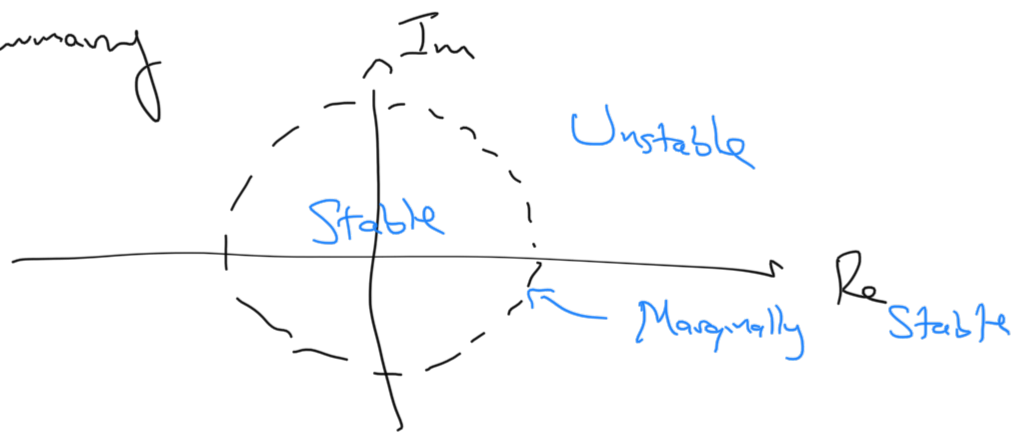
$\underbrace{\alpha z_n[k] + b_{n-1} u[k]}_{\text{call this } \tilde{b}_{n-1} \tilde{u}_{n-1}[k]}$

$$\Rightarrow \quad z_{n-1}[k+1] = \lambda_{n-1} z_{n-1}[k] + \tilde{b}_{n-1} \tilde{u}_{n-1}[k]$$

So if $|\lambda_{n-1}| < 1$, $n-1$ th row is also bounded!

Keep going recursively, conclude that this is also stable if $|\lambda_i| < 1 \quad \forall i$

- Summary



Stable if $|\lambda_i| < 1 \quad \forall i$

Unstable if $\exists \lambda_i$ s.t. $|\lambda_i| > 1$

Marginally Stable if $|\lambda_i| \leq 1 \quad \forall i$ and $\exists |\lambda_j| = 1$
(unstable in BIBO sense)

II. Continuous Time Stability

$$\frac{dx}{dt} = \lambda x + bu$$

$$x(t) = \ldots + \int^{t} e^{\lambda(t-\tau)} u(\tau)\, d\tau$$

$$x(t) = e^{\lambda t} x(0) + b \int_0^t e$$

- $Re\{\lambda\} > 0$, unstable

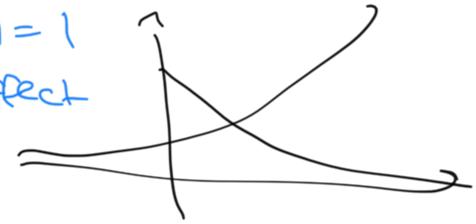
$\lambda = \sigma + j\omega$, where $\sigma = Re\{\lambda\}$ and $\omega = Im\{\lambda\}$

Pick $u = 0$

$$x(t) = e^{\lambda t} x(0) = e^{\sigma t} e^{j\omega t} x(0)$$

$$|x(t)| = |e^{\sigma t}|\, x(0) = e^{\sigma t} x(0)$$

$|e^{j\omega t}| = 1$ doesn't affect stability



$$\lim_{t\to\infty} |x(t)| \longrightarrow \infty \quad \text{if } x(0) \neq 0$$

Thus, if $Re\{\lambda\} = \sigma > 0$, then unstable

- $Re\{\lambda\} < 0$, stable

$$x(t) = \underbrace{e^{\lambda t} x(0)}_{S_1} + \underbrace{b \int_0^t e^{\lambda(t-\tau)} u(\tau)\, d\tau}_{S_2}$$

$$\lim_{t\to\infty} |S_1| = 0$$

$$\lim_{t\to\infty} |S_2| = ?$$

bounded by some B

$$\left| b \int_0^t e^{\lambda(t-\tau)} u(\tau)\, d\tau \right| \le \underbrace{|b| B}_{B'} \int_0^t |e^{\lambda(t-\tau)}|\, d\tau$$

$$|e^{\sigma(t-\tau)} \cancel{e^{j\omega(t-\tau)}}|$$

$$= B' \int_0^t |e^{\sigma(t-\tau)}|\, d\tau$$

$$= B' \int_0^t e^{\sigma(t-\tau)}\, d\tau$$

$$= B' e^{\sigma t} e^{-\sigma\tau} \Big|_0^t = B' e^{\sigma t} \frac{e^{-\sigma t} - 1}{}$$

$$B\, e \; \frac{\quad}{-\sigma}\Big|_0 \qquad\qquad -\sigma$$

$$= B' \frac{(1-e^{\sigma t})}{-\sigma} = B' \frac{e^{\sigma t}-1}{\sigma}$$

$$\lim_{t\to\infty} |S_2| \le \lim_{t\to\infty} B' \frac{e^{\sigma t}-1}{\sigma} = B'\left(-\frac{1}{\sigma}\right) < \infty$$

(Remember $\sigma < 0$)

Bounded! if $Re\{\lambda\} < 0$

- $Re\{\lambda\} = 0$

If no input,

$$|x(t)| = e^{\sigma t}|x(0)| = |x(0)|$$

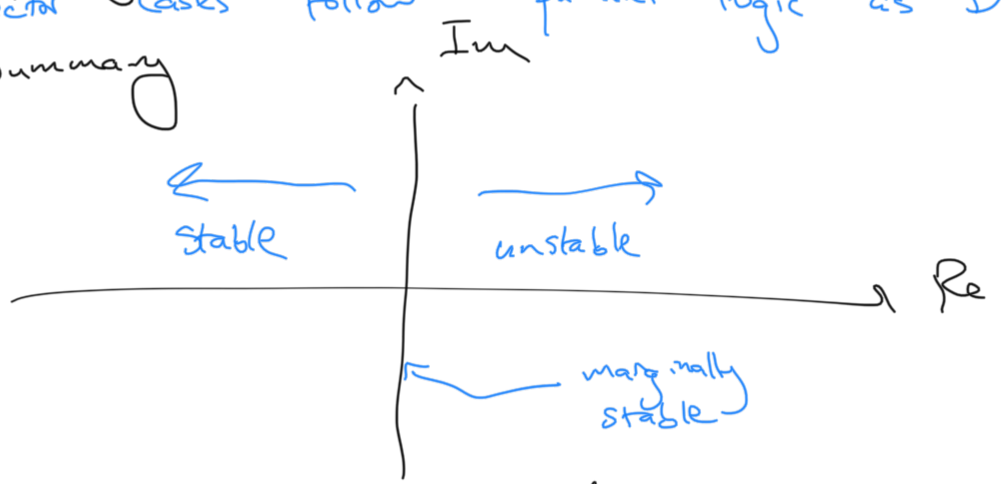
But can easily add input to make unbounded: $u = 1$

$$b\int_0^t u(\tau)\,d\tau = b\int_0^t d\tau = bt\Big|_0^t \longrightarrow \infty \text{ as } t\to\infty$$

Marginally stable (but unstable in the BIBO sense)

* Vector cases follow parallel logic as DT case

- Summary

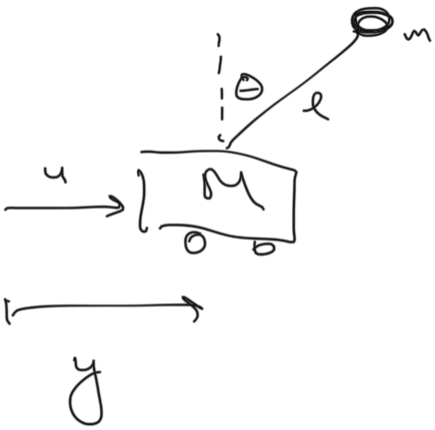


- Stable if $Re\{\lambda_i\} < 0 \quad \forall i$
- Unstable if $\exists \lambda_i$ s.t. $Re\{\lambda_i\} > 0$
- Marginally stable if $Re\{\lambda_i\} \le 0 \quad \forall i$ and $\exists \lambda_j$ s.t.

unstable in the ...

(unstable in the BIBO sense) $\mathrm{Re}\{\lambda_i\} = 0$

Ex: Inverted Pendulum on a Rolling Cart



Linearize about

$x_1 = \theta = 0$

$x_2 = \dot{\theta} = 0$

$x_3 = \dot{y} = 0$

$u = 0$

From HW4,

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{\ell M} g & 0 & 0 \\ -\frac{m}{M} g & 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ -\frac{1}{\ell M} \\ \frac{1}{M} \end{bmatrix} u$$

- CT system
- To check stability, calculate λ_i

$$\det \begin{vmatrix} -\lambda & 1 & 0 \\ \frac{M+m}{\ell M} g & -\lambda & 0 \\ -\frac{m}{M} g & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} \frac{M+m}{\ell M} g & 0 \\ -\frac{m}{M} g & -\lambda \end{vmatrix}$$

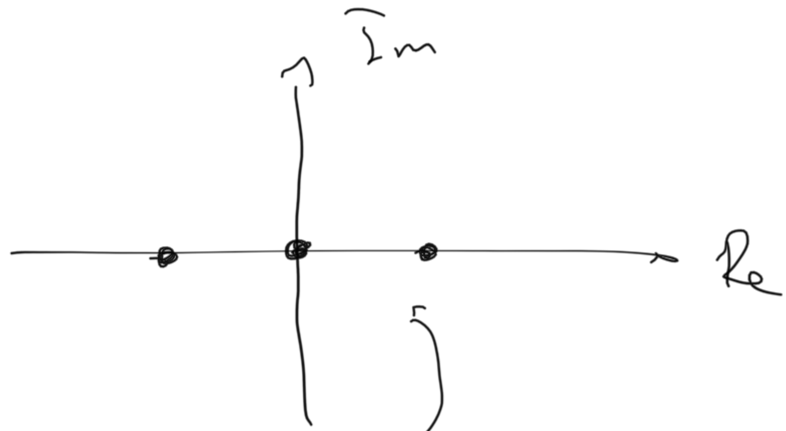
$$\cancel{+ 0 \begin{vmatrix} \frac{M+m}{\ell M} g & -\lambda \\ -\frac{m}{M} g & 0 \end{vmatrix}}$$

$$= -\lambda^3 - \left(\frac{M+m}{\ell M} g (-\lambda) - 0 \right)$$

$= -\lambda^3 + \lambda \frac{}{\ell M} g = 0$

$\lambda \left(\lambda^2 - \frac{M+m}{\ell M} g \right) = 0$

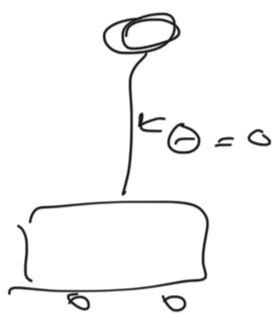
$\lambda = 0, \quad \pm\sqrt{\frac{M+m}{\ell M} g}$



$\boxed{\text{Unstable}}$

Physical Intuition:

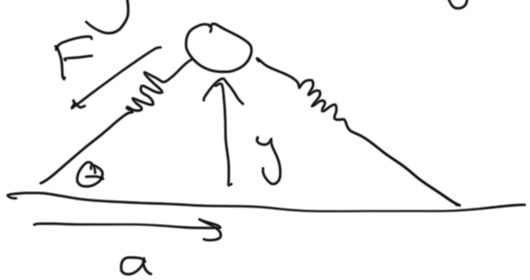
$\Theta = 0 \implies$



$\underbrace{\text{Unstable equilibrium}}$

Worksheet

② Stability in CT System



$X_0 < a$

Linearized about $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1-\frac{x_0}{a}\right) & 0 \end{bmatrix} x$$

Stable?

$$\det \begin{bmatrix} -\lambda & 1 \\ -\frac{2k}{m}\left(1-\frac{x_0}{a}\right) & -\lambda \end{bmatrix} = \lambda^2 + \frac{2k}{m}\left(1-\frac{x_0}{a}\right)$$

Difference of squares!

$$(\lambda+ja)(\lambda-ja) = \lambda^2 - j^2a^2 = \lambda^2+a^2$$

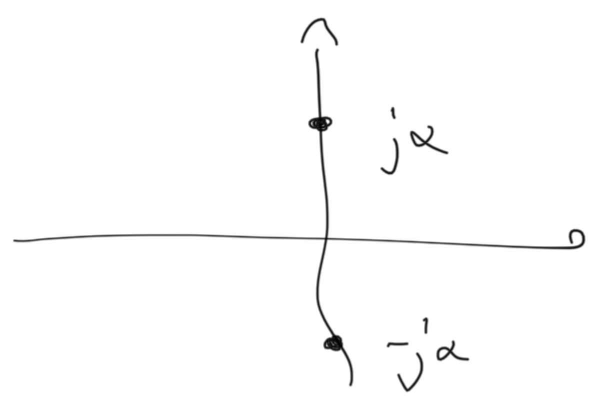
$$= \left(\lambda + j\sqrt{\frac{2k}{m}\left(1-\frac{x_0}{a}\right)}\right)\left(\lambda - j\sqrt{\frac{2k}{m}\left(1-\frac{x_0}{a}\right)}\right) = 0$$

Note that the stuff inside the sqrt is always nonnegative:

$$x_0 < a \implies 1-\frac{x_0}{a} > 0$$

$$k, m > 0$$

$$\lambda = \pm j\sqrt{\frac{2k}{m}\left(1-\frac{x_0}{a}\right)}$$



$$\operatorname{Re}\{\lambda\} = 0$$

$\implies$ $\boxed{\text{marginally stable}}$

but [illegible] nonzero bounded inputs)

(blows up if you put in a certain)

unstable w.r.t. to bounded input-bounded output stability

LC tank: $\lambda = \pm j \frac{1}{\sqrt{LC}}$ (?)

(3) Stability in DT Systems

$\alpha, \beta \in \mathbb{R}$, $b \neq 0$

a) $x[t+1] = \alpha x[t] + b u[t]$

$\lambda = \alpha$

scalar case: $\boxed{|\alpha| < 1}$

DT!!!

b) $\vec{x}[t+1] = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \vec{x}[t] + b\vec{u}[t]$

$$\begin{vmatrix} \alpha-\lambda & -\beta \\ \beta & \alpha-\lambda \end{vmatrix} = \overbrace{(\alpha-\lambda)^2}^{a^2} + \overbrace{\beta^2}^{b^2}$$

$$= (a+jb)(a-jb)$$

$$= (\alpha-\lambda+jb)(\alpha-\lambda-jb)$$

$$= 0$$

$\lambda = \overbrace{\alpha+j\beta}^{\lambda_1}, \overbrace{\alpha-j\beta}^{\lambda_2}$

Stable if $|\lambda_i| < 1$

$|\lambda_1| = \sqrt{\alpha^2+\beta^2}$

$|\lambda_2| = \sqrt{\alpha^2+(-\beta)^2} = \sqrt{\alpha^2+\beta^2}$

$|\lambda_1| = |\lambda_2| < 1$

$\Longrightarrow \boxed{\text{Pick } \alpha, \beta \text{ s.t. } \sqrt{\alpha^2 + \beta^2} < 1}$

c) $\vec{x}[t+1] = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \vec{x}[t] + b\vec{u}[t]$

$\boxed{\lambda = 1, 1}$

Upper triangular matrix; eigenvalues lie on the diagonal

independent of α

$\boxed{\text{Always marginally stable}}$

(unstable wrt to BIBO stability)