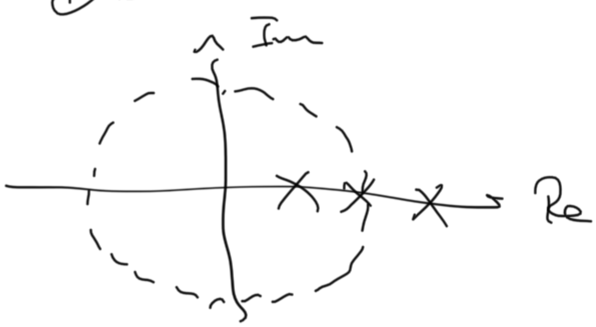
Discussion 6C

Transient Behavior and Feedback Control

* Transients – Predicting Behavior from Eigenvalues
  - DT
  - CT
* Feedback Control

I. Transients

a) Discrete Time



What do we expect?

From stability analysis,

$|\lambda| < 1 \longrightarrow$ stable, decays to 0

$|\lambda| = 1 \longrightarrow$ marginally stable: w/ 0 input, stays the same place

$|\lambda| > 1 \longrightarrow$ unstable, "blows up"

Consider the case $u = 0$ (free response)

$$x[k+1] = \lambda x[k] + \cancel{bu[k]}$$

$$\downarrow$$

$$x[k] = \lambda^k x[0]$$

In polar form, $\lambda = |\lambda| e^{j\omega}$

Then,

$$\boxed{x[k] = \lambda^k x[0] = \underbrace{|\lambda|^k}_{\text{envelope}} \underbrace{e^{j\omega k}}_{\text{oscillations}} x[0]}$$

Euler's Formula:

$$x[k] = |\lambda|^k x[0] \left(\cos(\omega k) + j\sin(\omega k) \right)$$

Refine intuition?

$|\lambda| < 1 \longrightarrow$ envelope decays to 0

$|\lambda| = 1 \longrightarrow$ constant envelope

$|\lambda| > 1 \longrightarrow$ envelope blows up

For plotting purposes look at real part

$$|\lambda|^k x[0] \cos(\omega k)$$

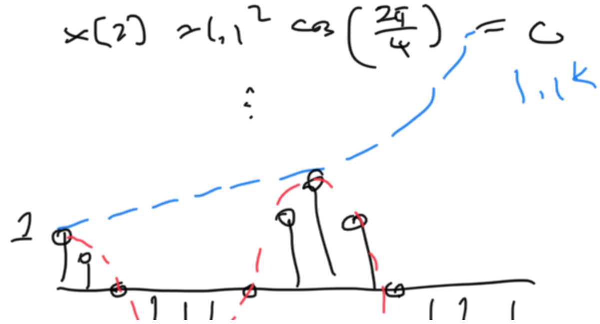
Ex: $\omega = \frac{\pi}{4}$, $|\lambda| = 1.1$, $x[0] = 1$

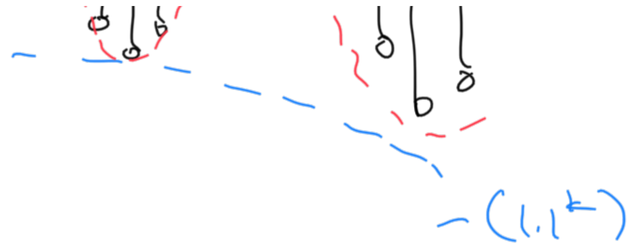
$$x[0] = 1$$

$$x[1] = 1.1 \cos\left(\frac{\pi}{4}\right) \approx 0.78$$

$$x[2] = 1.1^2 \cos\left(\frac{2\pi}{4}\right) = 0$$

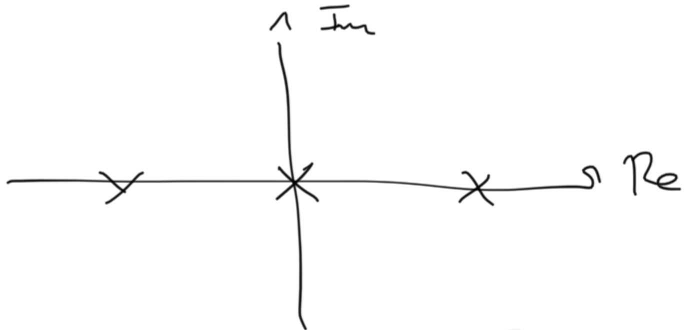
$$\vdots$$





Look at this more w/ Q1 of the discussion

b) Continuous Time



What do we expect?

$Re\{\lambda\} < 0 \longrightarrow$ stable, decays to 0

$Re\{\lambda\} = 0 \longrightarrow$ marginally stable: w/ 0 input, should stay "at the same place"

$Re\{\lambda\} > 0 \longrightarrow$ unstable, blows up

Same thing: $u = 0$

$$\frac{dx}{dt} = \lambda x(t)$$

$$x(t) = x(0) e^{\lambda t}$$

Let's look at λ in Cartesian form:

$$\lambda = \sigma + j\omega \; ; \quad \sigma, \omega \in \mathbb{R}$$

$$\boxed{x(t) = x(0) e^{\sigma t} e^{j\omega t}}$$

envelope $\qquad$ oscillations

Euler's formula,

$$x(t) = x(0)e^{\sigma t}\left(\cos(\omega t) + j\sin(\omega t)\right)$$

Refine intuition:

$Re\{\lambda\} < 0 \longrightarrow$ exponentially decaying envelope

$Re\{\lambda\} = 0 \longrightarrow$ constant envelope

$Re\{\lambda\} > 0 \longrightarrow$ exp. growing envelope

Ex: $\sigma = \ln 1.1$

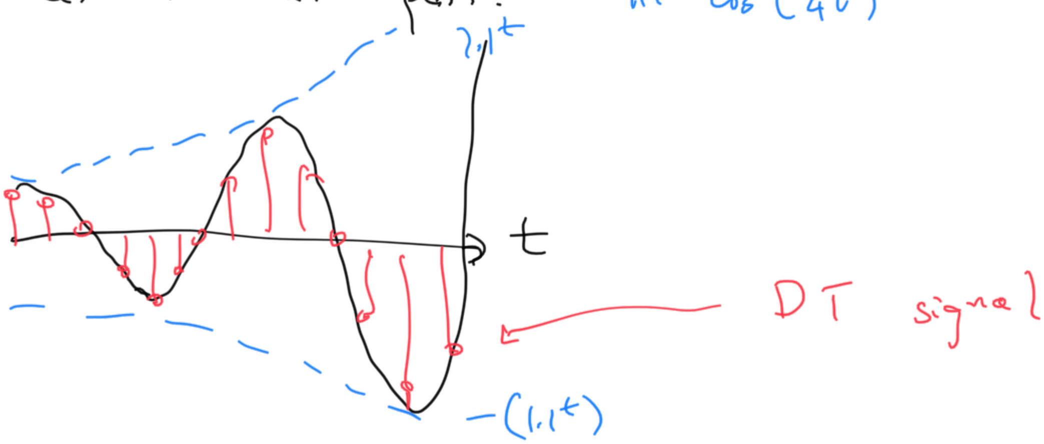
$\omega = \frac{\pi}{4}$

$x(0) = 1$

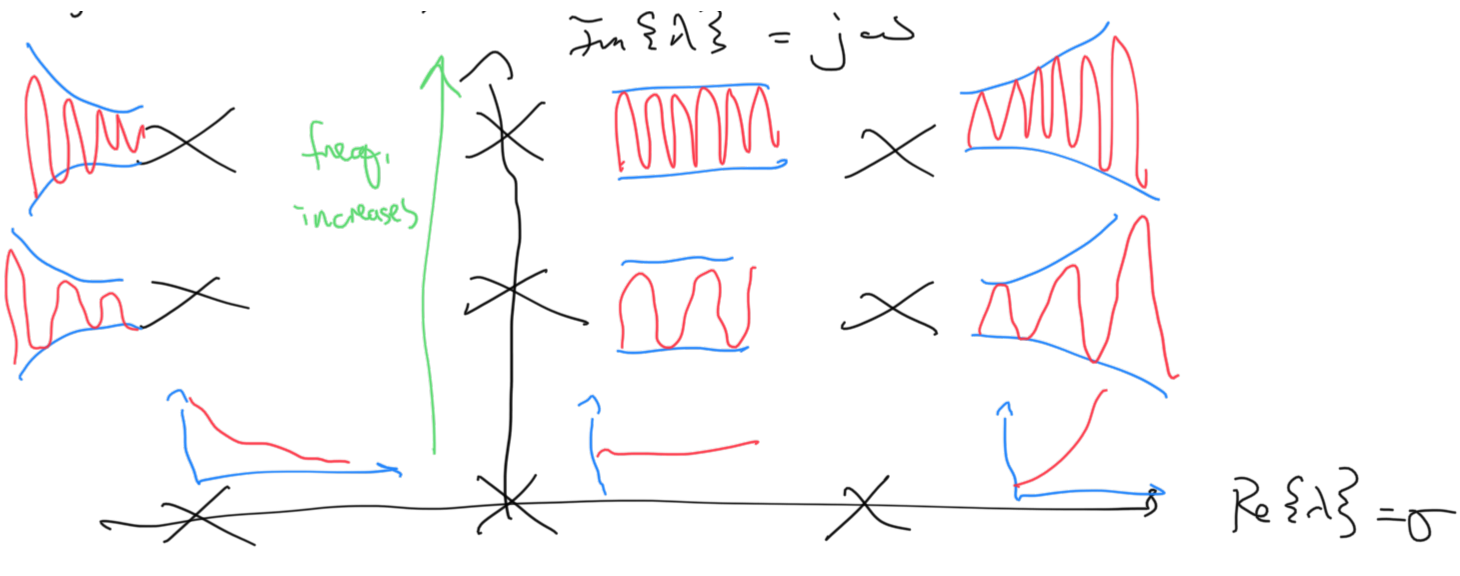
Then, $x(t) = e^{(\ln 1.1)t} e^{j\frac{\pi}{4}t}$

$= 1.1^t e^{j\frac{\pi}{4}t}$

Look at the real part: $1.1^t \cos\left(\frac{\pi}{4}t\right)$

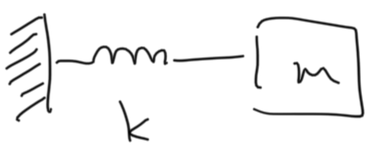


For an extensive plot of example transient waveforms for each possible eigenvalue location, see lec12.pdf notes



envelope changes from exp. decay to exp. growth

Ex: Damped Harmonic Oscillator



$F = m\ddot{x} = -kx$ (no damping)

Add damping force $-\gamma\dot{x}$

$$\Rightarrow F = m\ddot{x} = -kx - \gamma\dot{x}$$

$$\frac{d^2x}{dt^2} = -\underbrace{\frac{k}{m}}_{\omega_0^2}x - \frac{\gamma}{m}\dot{x} \qquad \left(\omega_0 = \sqrt{\frac{k}{m}}\right)$$

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \frac{\gamma}{m}\dot{x}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\frac{\gamma}{m} \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Find eigenvalues to predict how component values affect transient behavior:

$$\det\begin{bmatrix} -\lambda & 1 \\ -\omega_0^2 & -\lambda-\frac{\gamma}{m} \end{bmatrix} = \lambda\left(\lambda+\frac{\gamma}{m}\right)+\omega_0^2 = 0$$

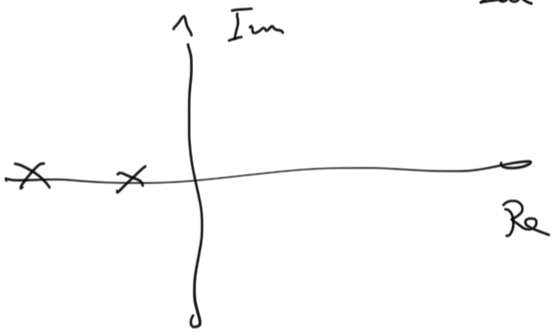
Quadratic formula:

$$\lambda = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \omega_0^2}$$

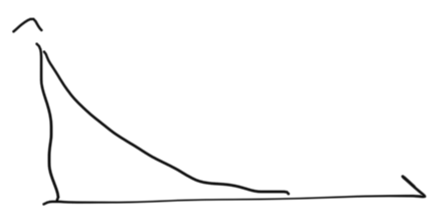
$\gamma, m, \omega_0 > 0$

If $\frac{\gamma}{2m} > \omega_0$, $\sqrt{\left(\frac{\gamma}{2m}\right)^2 - \omega_0^2} < \frac{\gamma}{2m}$ ←

$\Longrightarrow \lambda = -\frac{\gamma}{2m} \pm \sqrt{} < 0$, real-valued



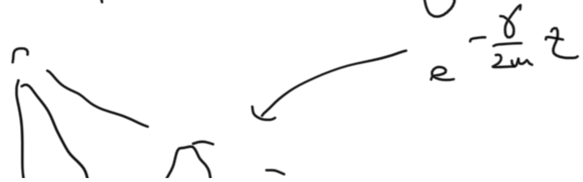
overdamped response



If $\frac{\gamma}{2m} > \omega_0$, $\sqrt{\left(\frac{\gamma}{2m}\right)^2 - \omega_0^2} = j\sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$

$\lambda = -\frac{\gamma}{2m} \pm j\sqrt{}$

$\Longrightarrow$ oscillations w/ exponential decay



underdamped response

Look familiar?

RLC ckt

$$\lambda = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \omega_0^2}$$

$$\omega_0^2 = \frac{1}{LC}$$

Compare to:

$$\lambda = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \omega_0^2}$$

$L \longleftrightarrow m$

$R \longleftrightarrow \gamma$

$C \longleftrightarrow 1/k$

Mechanical - Electrical Analogy

II. Feedback Control

Op-Amps:

Similarly, feed output back into the input

$$\vec{x}[k+1] = A\vec{x}[k] + B\vec{u}[k]$$

"Feedback control policy": $\boxed{u[k] = -K\vec{x}[k]}$

$$\vec{x}[k+1] = A\vec{x}[k] - BK\vec{x}[k]$$

$$\boxed{\vec{x}[k+1] = \underbrace{(A-BK)}_{A_{cl}}\vec{x}[k]}$$

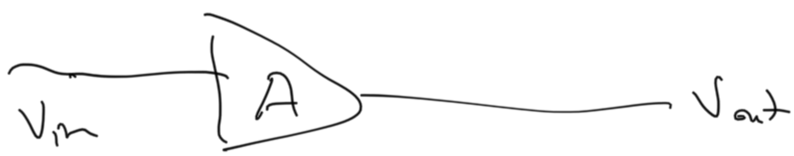
- Use for eigenvalue placement

> If controllable, guaranteed that you can place eigenvalues anywhere w/ appropriate K

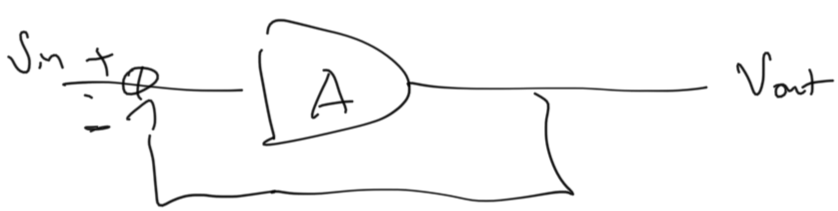
Q: open-loop vs closed-loop control?

Open-loop: $u[k] \neq f(\vec{x}[k])$

Op-Amp:



open-loop



closed-loop

Open loop control: Pick $u(0), u(1) \ldots u(k-1)$

S.T. $\vec{x}[k] = A^k\vec{x}[0] + A^{k-1}B\vec{u}[0]$

$+ \ \ldots$

$+ AB\vec{u}[k-2]$

$+ B\vec{u}[k-1]$

$u[i]$ independent of outputs at each time step

Closed loop control: $u[i]$ dependent on outputs at each time step

Open loop $\neq$ Optimal control

$\hookrightarrow$ min energy

weighted min energy

- How to assign k values?

$$\boxed{\text{Match coefficients of desired char. poly. } \chi \text{ to } \chi_{A_{cl}}(k)}$$

Ex: 2×2

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = [K_1 \ K_2]$$

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ 2-k_1 & 3-k_2 \end{bmatrix}$$

$$\chi = \Big| -\lambda \qquad 1 \Big|$$

A_d $\begin{vmatrix} 2-k_1 & 3-k_2-\lambda \end{vmatrix}$

$$= \lambda(\lambda-3+k_2) - (2-k_1)$$

$$= \lambda^2 + \lambda(K_2-3) + (K_1-2) \leftarrow$$

$$\chi = (\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2 \leftarrow$$

$\Rightarrow$ Solve
$$(-\lambda_1-\lambda_2) = K_2-3$$
$$\lambda_1\lambda_2 = K_1-2$$

Worksheet!

② Eigenvalue Placement in DT

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] + \vec{w}[t]$$

a) Controllable?

$$\mathcal{C} = [B \quad AB] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \leftarrow \boxed{\text{full rank, controllable!}}$$

if not full rank, then $\det \mathcal{C} = 0$

$$\det \mathcal{C} = 2 \neq 0 \longrightarrow \text{full rank}$$

b) $$\det\begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} = \lambda(\lambda+1) - 2 = \lambda^2 + \lambda - 2$$

$$= (\lambda+2)(\lambda-1)$$

$\Rightarrow \quad \lambda = -2, 1$

$\rightarrow \quad \lambda = -2, \; 1$

$\qquad\qquad \lambda_1 \quad \lambda_2$

$$\boxed{\text{Not stable, b/c } |\lambda_1| > 1 \quad (DT)}$$

c) Careful: $u[t] = +K\vec{x}[t] = [k_1 \;\; k_2]\,\vec{x}[t]$

$$A_{cl} = A + BK$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \;\; k_2]$$

$$\begin{matrix} k_1 & k_2 \\ 0 & 0 \end{matrix}$$

Thus, $\boxed{\vec{x}[t+1] = A_{cl}\,\vec{x}[t] = \begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix} \vec{x}[t]}$

d) Want $\lambda_1 = -\frac{1}{2}, \; \lambda_2 = \frac{1}{2}$

Match characteristic polynomials:

$$\chi = (\lambda - \tfrac{1}{2})(\lambda + \tfrac{1}{2}) = \lambda^2 - \tfrac{1}{4}$$

$$\det A_{cl} = \chi_{A_{cl}} = \det \begin{bmatrix} k_1 - \lambda & 1+k_2 \\ 2 & -1-\lambda \end{bmatrix}$$

$$= \lambda^2 + (1-k_1)\lambda - (k_1 + 2k_2 + 2)$$

$$= \lambda^2 + 0\lambda - \frac{1}{4}$$

Solve

$1 - k_1 = 0 \longrightarrow k_1 = \frac{1}{4}$

$k_1 + 2k_2 + 2 = \frac{1}{4} \longrightarrow 2k_2 + 3 = \frac{1}{?}$

$k_2 = -\frac{11}{8}$, $k_1 = 1$

$$\boxed{K = \begin{bmatrix} 1 & -\frac{11}{8} \end{bmatrix}}$$

e) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] \longrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t]$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

Controllable?

$$\mathcal{C} = [B \quad AB] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \boxed{\begin{array}{c}\text{rank } 1 \\ \text{not controllable}\end{array}}$$

f) $A_{cl} = A + BK = A + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [k_1 \ \ k2]$ $\begin{matrix} k_1 & k_2 \\ k_1 & k_2 \end{matrix}$

$$= \begin{bmatrix} k_1 & 1+k_2 \\ 2+k_1 & -1+k_2 \end{bmatrix}$$

$$\det \underbrace{\begin{bmatrix} k_1-\lambda & 1+k_2 \\ 2+k_1 & -1+k_2-\lambda \end{bmatrix}}_{A_{cl}-\lambda I} = (k_1-\lambda)(k_2-1-\lambda) - (k_1+2)(k_2+1)$$

$$= \cancel{k_1k_2} - k_1 - \underline{k_1\lambda} - \underline{k_2\lambda} + \underline{\lambda} + \lambda^2 - \cancel{k_1k_2} - k_1 - 2k_2 - 2$$

$$= \lambda^2 + (1-k_1-k_2)\lambda - 2(1+k_1+k_2)$$

Note that $-(1+k_1+k_2) \times 2 = -2(1+k_1+k_2)$

$-(1+k_1+k_2)+2 = 1-k_1-k_2$

$$= (\lambda+2)(\lambda-(1+k_1+k_2)) = 0$$

$$\boxed{\lambda_1 = -2, \quad \lambda_2 = 1+k_1+k_2}$$

Fixed: $|\lambda_1| > 1 \implies$ unstable

Will go over Q1, Q3 tomorrow