Discussion 6D

MT2 Review

- Linearization
- Discretization
- Stability
- Feedback Control (up to Tuesday's lecture: no CCF, or tracking control)

I. Linearization

Everything comes from Taylor expansions

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x^*)}{n!} (x-x^*)^n$$

To linearize, take terms up to order I

$$\boxed{\begin{aligned} f(x) &\simeq f(x^*) + f'(x^*)(x-x^*) \\ \delta f &= f(x) - f(x^*) \\ \delta x &= x - x^* \end{aligned}}$$

Scalar case

In general, vector case

$$\boxed{f(\vec{x}) \simeq f(\vec{x}^*) + \nabla f \Big|_{\vec{x}=\vec{x}^*} (\vec{x} - \vec{x}^*)}$$

where ∇f denotes the Jacobian

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$\vec{x} - \vec{x}^* = \begin{bmatrix} x_1 - x_1^* \\ \vdots \\ x_n - x_n^* \end{bmatrix}$$

For some state space model, we have:

$$\frac{d\vec{x}}{dt} = f(\vec{x}, \vec{u}) \qquad CT$$

$$\text{or} \quad \vec{x}[t+1] = f(\vec{x}[t], \vec{u}[t]) \qquad DT$$

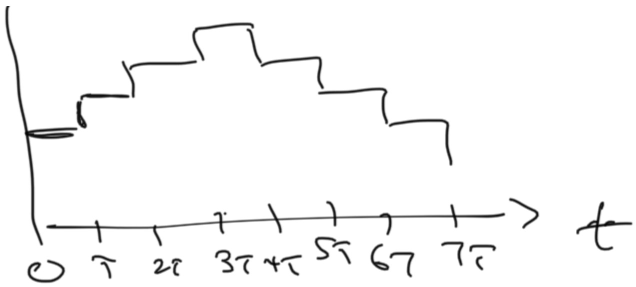
$$\boxed{f(\vec{x}, \vec{u}) \simeq f(\vec{x}^*, \vec{u}^*) + \underbrace{\nabla_x f \Big|_{\vec{x}^*, \vec{u}^*}}_{A} (\vec{x} - \vec{x}^*) + \underbrace{\nabla_u f \Big|_{\vec{x}^*, \vec{u}^*}}_{B} (\vec{u} - \vec{u}^*)}$$

$$Ax + Bu$$

II. Discretization

Idea: use <u>digital control</u>, which in CT we represent as <u>piecewise constant inputs</u>

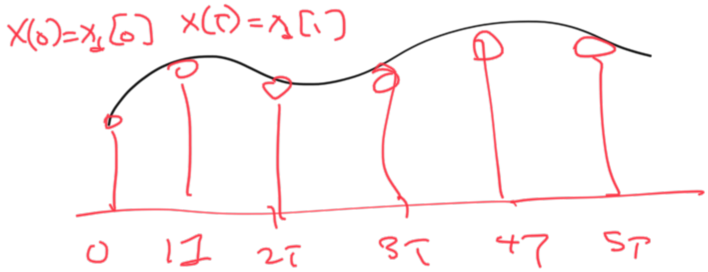
$\uparrow u(t)$



Notation: $x_d[k] = x(kT) \longleftarrow$ CT

$\uparrow$ DT

$u_d[k] = u(kT)$



Method: solve differential equation

$$\frac{dx}{dt} = \lambda x(t) + b u(t)$$

or $$\frac{d\vec{x}}{dt} = A\vec{x}(t) + B\vec{u}(t)$$

with a constant input

in the interval $t \in [kT, (k+1)T)$

In this time interval, $u(t) = u_d[k] = \text{const.}$

Using techniques from the 1st half of class:

Scalar case

$$x(t) = e^{\lambda(t-kT)} x(kT) + b u_d[k] \int_0^T e^{-\lambda s} ds$$

P... $t = (k+1)T$

Plug in $t = (k+1)T$

$$x((k+1)T) = e^{\lambda T} x(kT) + b u_d[k] \int_0^T e^{\lambda s} ds$$

$$\boxed{x_d[k+1] = \underbrace{e^{\lambda T}}_{\lambda_d} x_d[k] + \underbrace{\left(b\int_0^T e^{\lambda s} ds\right)}_{b_d} u_d[k]}$$

- Vector case

$\hookrightarrow$ Try to turn into a system of independent scalar case

a) A is diagonalizable

$$\frac{d\vec{x}}{dt} = A\vec{x} + B\vec{u}$$

$$\vec{z} = V^{-1}\vec{x}$$

$$\frac{d\vec{z}}{dt} = \Lambda\vec{z} + V^{-1}B\vec{u}$$

After even more math:

$$\vec{z}_d[k+1] = e^{\Lambda T}\vec{z}_d[k] + \Lambda_T V^{-1} B \vec{u}_d[k]$$

Go back to original basis, convert w/ $\vec{x} = V\vec{z}$

$$\vec{x}_d[k] = V e^{\Lambda T} V^{-1} \vec{x}_d[k] + V\Lambda_T V^{-1} B \vec{u}_d[k]$$

$$\underbrace{\qquad\qquad}_{A_d} \qquad \underbrace{\qquad\qquad}_{B_d}$$

where
$$e^{\Lambda T} = \begin{bmatrix} e^{\lambda_1 T} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n T} \end{bmatrix}$$

$$\Lambda_T = \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & 0 \\ & \ddots & \\ 0 & & \int_0^T e^{\lambda_n s} ds \end{bmatrix}$$

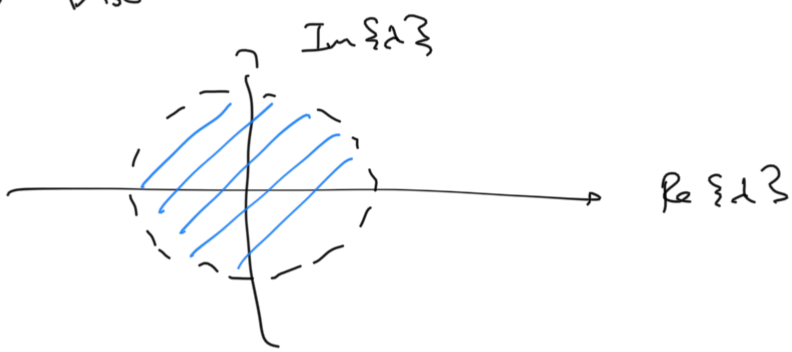
b) A is not diagonalizable

- matrix exponential (out-of-scope)
- solve $\underbrace{\text{diff eq w/ const inputs}}_{}$ using other means if system is simple enough

(e.g. our model: used FTC and directly integrated)

III. Stability

a) Discrete Time



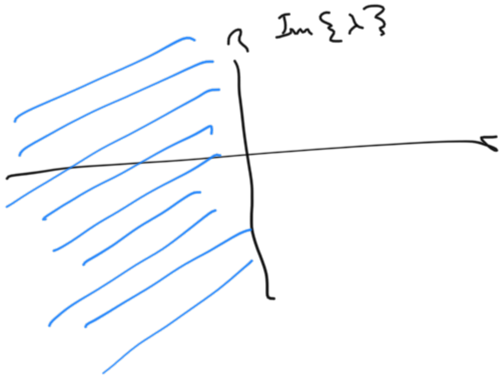
Stable: $|\lambda_i| < 1 \quad \forall i$

unstable: $\exists \lambda_i$ s.t. $|\lambda_i| > 1$

...lly stable? $|\lambda_i| \le 1 \ \forall i$ and $\exists \lambda_i$ s.t. $|\lambda_i| = 1$

marginally

b) Continuous Time



stable: $\mathrm{Re}\{\lambda_i\} < 0 \quad \forall i$

unstable: $\exists\, \lambda_i$ s.t. $\mathrm{Re}\{\lambda_i\} > 0$

marginally stable: $\mathrm{Re}\{\lambda_i\} \leq 0 \quad \forall i$ and $\exists\, |\lambda_j|$ s.t. $\mathrm{Re}\{\lambda_j\} = 0$

Added note on marginal stability:

$$x(t) = x_0 e^{\lambda t} + b\int_0^t e^{\lambda(t-\tau)} u(\tau)\, d\tau$$

Input $u(t) = e^{j\omega_0 t}$ where $\lambda = j\omega_0$

(so match the frequency of the eigenvalue)

$\Longrightarrow$ pure resonance

bounded input, unbounded output

$$x(t) = x_0 e^{j\omega_0 t} + b\int e^{j\omega_0 t} \cancel{e^{-j\omega_0 \tau}}\, e^{j\omega_0 \tau}\, d\tau$$

$$= \text{''} \quad + b e^{j\omega_0 t} \int^t d\tau$$

$$\lim_{t \to \infty} x(t) \to \infty$$

III. Feedback Control

open-loop: $u[i]$ <u>independent</u> of outputs

closed-loop: $u[i]$ <u>dependent</u> on outputs,

i.e. $u[k] = f(\vec{x}[k])$

Typically, linear control:

$$\boxed{\underset{m \times 1}{\vec{u}[k]} = -\underset{m \times n}{K}\,\underset{n \times 1}{\vec{x}[k]}}$$

Then,

$$\begin{aligned} \vec{x}[k+1] &= A\vec{x}[k] + Bu[k] \\ &= A\vec{x}[k] - BK\vec{x}[k] \\ &= \underbrace{(A-BK)}_{A_{cl}}\vec{x}[k] \end{aligned}$$

If system is controllable, guaranteed we can place the eigenvalues anywhere (CCF)

not on Tues lecture, ergo it's not in

Choose eigenvalues and get some desired char. poly: χ_{goal}

Scope for MT2

⇒ $\boxed{\text{match } \chi_{goal} \text{ to } \chi_{A_{cl}}}$

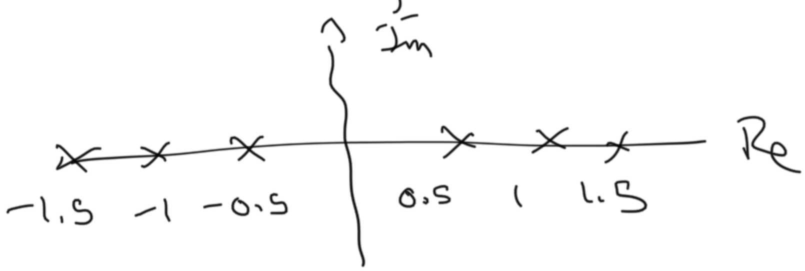
function of K

Match coefficients to choose K values

(1) Dis GC, Q1 — DT System Response

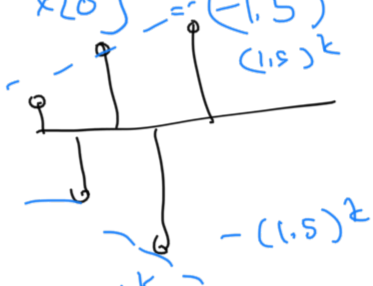
$x[k+1] = \lambda x[k]$

$x[0] = 1$



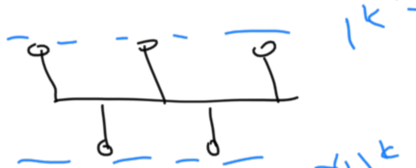
(i) $\lambda = -1.5$

$x[k] = \lambda^k x[0] = (-1.5)^k x[0] = (-1.5)^k$



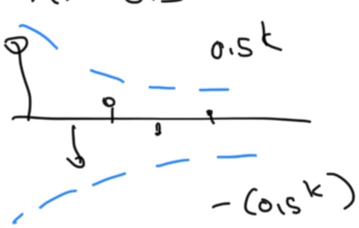
$\boxed{\text{unstable, b/c } |-1.5| > 1}$

(ii) $\lambda = -1$



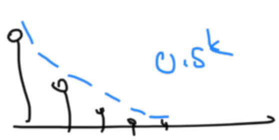
$\boxed{\text{marginally stable}}$

(iii) $\lambda = -0.5$



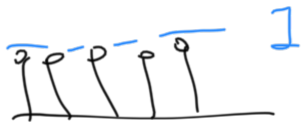
$\boxed{\text{stable, b/c } |-0.5| < 1}$

(iv) $\lambda = 0.5$



$\boxed{\text{stable, b/c } |\lambda| < 1}$

iv) $\lambda = 1$



$\boxed{\text{marginally stable}}$

$x[0] = 1$

$\lambda = j$

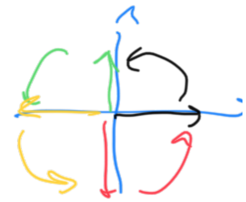
$x[k] = j^k x[0] = j^k$

$x[0] = 1$

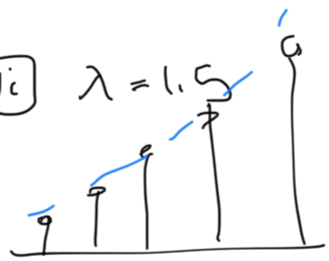
$x[1] = j$

$x[2] = -1$

$x[3] = -j$



vi) $\lambda = 1.5$



$\boxed{\text{unstable, b/c } |\lambda| > 1}$

② Dis 6C, Q3 — Eigenvalue Placement ~~in CCF~~

Just real part:

$$\vec{x}[t+1] = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & -6 \end{bmatrix}}_{A} \vec{x}[t] + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} \vec{u}[t]$$

Recall that

$|\lambda|^k e^{j\omega k}$

$j = 1 e^{j\frac{\pi}{2}} \to e^{j\frac{\pi}{2}k}$

Real part:

$x[k] = \cos\left(\frac{\pi}{2}k\right)$

a)

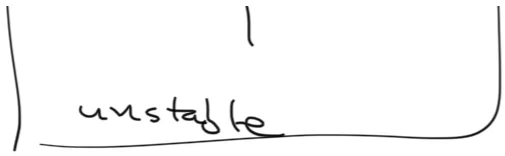
$$\det(A - \lambda I) = \det \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -9 & -6-\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -9 & -6-\lambda \end{vmatrix} - 1 \cancel{\begin{vmatrix} 0 & 1 \\ 0 & -6-\lambda \end{vmatrix}} + 0 \cancel{\begin{vmatrix} 0 & -\lambda \\ 0 & -9 \end{vmatrix}}$$

$$= -\lambda(\lambda(\lambda+6) + 9) = -\lambda(\lambda^2 + 6\lambda + 9) = 0$$

$$\Rightarrow \lambda(\lambda+3)^2 = 0$$

$$\boxed{\lambda = 0, -3, -3}$$



b) $u[t] = Kx[t]$ ← Careful: not $-Kx[t]$

$$A_{cl} = A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_0 & -9+k_1 & -6+k_2 \end{bmatrix}$$

$$\det(A_{cl} - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ k_0 & -9+k_1 & -6+k_2-\lambda \end{vmatrix}$$

$$= -\lambda\left(-\lambda(-6+k_2-\lambda) - (-9+k_1)\right) - 1\left((0 - k_0)\right) + 0$$

$$= -\lambda\left(\lambda^2 - \lambda(-6+k_2) + (9-k_1)\right) + k_0$$

$$= -\lambda^3 + \lambda^2(-6+k_2) - \lambda(9-k_1) + k_0 = 0$$

Match to $\lambda = 0, 1/2, -1/2$

$$\chi_{goal} = (\lambda - 0)(\lambda - \tfrac{1}{2})(\lambda + \tfrac{1}{2}) = \lambda(\lambda^2 - \tfrac{1}{4})$$

$$= \lambda^3 - \frac{1}{4}\lambda$$

$$\chi_{A_{cl}} = \lambda^3 - \lambda^2(-6+k_2) + \lambda(9-k_1) - k_0 = 0$$

$$\begin{matrix} \parallel & & \parallel & & \parallel & & \parallel \\ 1 & & 0 & & -\frac{1}{\ } & & \end{matrix}$$

4 0

$-6+k_2=0 \implies k_2=6$

$9-k_1=-\frac{1}{4} \implies k_1=9+\frac{1}{4}=\frac{37}{4}$

$k_0=0$

$$\boxed{K=[k_0\ k_1\ k_2]=\left[0\ \ \frac{37}{4}\ \ 6\right]}$$

If we used $u[k]=-Kx[k]$

Then $K=\left[0\ \ -\frac{37}{4}\ \ -6\right]$ (Just flip by a minus sign)

"Placing eigenvalues"

A has some eigenvalues — potentially unstable

A_{cl} has eigenvalues that depend on K — has eigenvalues that depend on K

We choose K s.t. the eigenvalues of A_{cl} are where we want them to be
(design)
(e.g. make all $|\lambda_i|<1$ to make stable)

c) Consider the case that k_i are limited to $[-5,5]$

Can we make the system stable?

 ← very edge of where

all the λ could be and still be stable

Suppose $\lambda = -1$: then $\chi = (\lambda+1)^3$

$$= \lambda^3 + 3\lambda^2 + 3\lambda + 1$$

$\lambda = 1$: then $\chi = (\lambda - 1)^3$

$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1$$

$\lambda = -j$: then $\chi = (\lambda + j)^3 = \lambda^3 + 3j\lambda^2 - 3\lambda - j$

Then the magnitude of the coefficients has to be:

$$|c_0| = 1 \quad |c_1| = 3, \quad |c_2| = 3, \quad |c_3| = 1$$

$$\lambda^0 \qquad \lambda^1 \qquad \lambda^2 \qquad \lambda^3$$

Try $\lambda = -0.99$, $\qquad \chi = c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3$

then $(\lambda + 0.99)^3 = \lambda^3 + 2.97\lambda^2 + 2.9403x + 0.97\ldots$

$$|c_3| = 1 \qquad |c_2| < 3 \qquad |c_1| < 3 \qquad |c_0| < 1$$

Conditions on coefficients for λ to all be inside the unit circle

$\Rightarrow$ Compare to χ_{Acl}

$$\lambda^3 - (-6 + k_2)\lambda^2 - (-9 + k_1)\lambda - k_0$$

fine $\quad k_2 = 3,$ $\quad k_1 = 6,$

fine $\quad$ out of range!

$\boxed{\text{Can never make the system stable!}}$

(3) Fall 2016, MT 2, Q 2

Linearization + Stability + Equilibrium Points

$$x[t+1] = f(x[t])$$

$$f(x) = 2x - 2x^2$$

a) Equilibrium points:

b) Linearize about each eq. point, Stable?

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a) DT system:

equilibrium $\longrightarrow$ $x[t+1] = x[t]$ $\leftarrow$ constant solution

$$f(x^*) = x^*$$

$$\Rightarrow 2x^* - 2x^{*2} = x^*$$

$$x^* - 2x^{*2} = 0$$

$$x^*(1 - 2x^*) = 0$$

$$\boxed{x^* = 0,\ 0.5}$$

b) $x^* = 0$?

b) $x^* = 0$,

Scalar system!

$$f(x) \simeq f(x^*) + f'(x^*)(x - x^*)$$

$$f(x) \simeq f(0) + f'(0)(x)$$

$$f(0) = 2(0) - 2(0^2) = 0$$

$$f'(0) = (2 - 4x)\Big|_{x=0} = 2$$

Thus, $\boxed{f(x) \simeq 2x}$

Stable?

$$x[t+1] = f(x[t]) \simeq 2x[t] = \lambda x[t] + bu[t]$$

$$\boxed{\lambda = 2 > 1 \implies \text{unstable}}$$

$x^* = 0.5$

$$f(x) \simeq f(0.5) + f'(0.5)(x - 0.5)$$

$$f(0.5) = 2(0.5) - 2(0.5^2) = 0.5$$

$$f'(0.5) = 2 - 4(0.5) = 0$$

$$\boxed{f(x) = 0.5}$$

$$x[t+1] = f(x[t]) = 0.5 = \lambda x[t] + bu[t]$$

$\Rightarrow \boxed{\lambda = 0,}$

stable

(4) Dis CD, Q1, part b

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t)$$

$$\Longrightarrow \quad \vec{x}[k+1] = A_d \vec{x}[k] + B_d \vec{u}[k]$$

$$A_d = V e^{\Lambda T} V^{-1}$$

$$B_d = V \Lambda_T V^{-1} B$$

First confirm you have a diagonalizable system!

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda(\lambda+3) + 2$$

$$= \lambda^2 + 3\lambda + 2$$

$$= (\lambda+1)(\lambda+2)$$

Wouldn't work for car model, for example

$$\Longrightarrow \quad \lambda_1 = -2$$
$$\lambda_2 = -1$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longleftarrow$ not diagonalizable

$$A + 2I = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \longrightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$A + 1I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \longrightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$V^{-1} = \frac{1}{-1+2}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

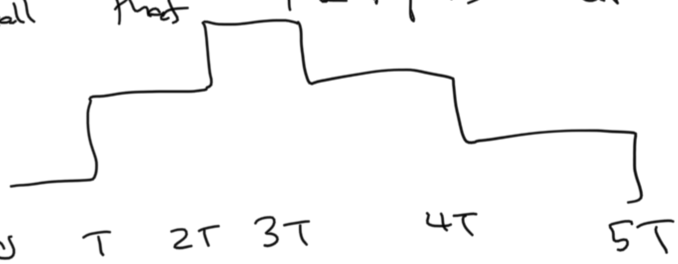
$$A_d = V e^{\Lambda T} V^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} e^{-2T} & 0 \\ 0 & e^{-T} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -e^{-2T} & -e^{-2T} \\ 2e^{-T} & e^{-T} \end{bmatrix}$$

$$\boxed{A_d = \begin{bmatrix} -e^{-2T}+2e^{-T} & -e^{-2T}+e^{-T} \\ -2e^{-2T}-2e^{-T} & 2e^{-2T}-e^{-T} \end{bmatrix}}$$

What is T?

Recall that the inputs are piecewise constant:



$$B_d = V \Lambda_T V^{-1} B$$

$$\Lambda_T = \begin{bmatrix} \int_0^T e^{\lambda_1 s}\,ds & 0 \\ 0 & \int_0^T e^{\lambda_2 s}\,ds \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^T e^{-2s}\,ds & 0 \\ 0 & \int_0^T e^{-s}\,ds \end{bmatrix}$$

$$= \begin{bmatrix} \left.\dfrac{e^{-2s}}{-2}\right|_0^T & 0 \\ 0 & \left.\dfrac{e^{-s}}{-1}\right|_0^T \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{e^{-2T}-1}{-2} & 0 \\ 0 & \dfrac{e^{-T}-1}{-1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{e^{-2T}-1}{-2} & 0 \\ 0 & \dfrac{e^{-T}-1}{-1} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{e^{-2T}-1}{-2} & 0 \\ 0 & \dfrac{e^{-T}-1}{-1} \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} e^{-2T}-1 \\ -2(e^{-T}-1) \end{bmatrix}$$

$$\Rightarrow \boxed{B_d = \begin{bmatrix} e^{-2T}-1-2(e^{-T}-1) \\ -2(e^{-2T}-1)+2(e^{-T}-1) \end{bmatrix} = \begin{bmatrix} 1-2e^{-T}+e^{-2T} \\ 2e^{-T}-2e^{-2T} \end{bmatrix}}$$