Discussion 7A

Beyond 16B?

Circuits Part II → EE105 : analog circuits, transistor ckts
→ EE130 : device physics
→ EE113, EE137A, EE134 : power

SVD, PCA → EECS 127 : optimization
→ EECS 189 : ML

control theory → EE 128 : feedback control
→ EE 221A : linear systems?
→ EECS 106A : robotics

DFT and some signal processing → EE120 ; signals and systems

Controller Canonical Form (CCF)

* Motivation

"Derivation" of Procedure

I. Motivation

Recall from feedback control: if a system is <u>controllable</u>, then we are <u>guaranteed</u> we can place the eigenvalues anywhere, with appropriate choice of K

a) How do we know that?

b) How can we choose K for more complicated systems?

Dis 6C:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \det \begin{bmatrix} k_1 - \lambda & k_2 + 1 \\ k_1 + 2 & k_2 - 1 - \lambda \end{bmatrix}$$

... (a lot of math)

$$\Rightarrow \lambda^2 + (1 - k_1 - k_2)\lambda - 2(1 + k_1 + k_2) = 0$$

Still doable by hand, but we can imagine that for larger systems, the eqns get uglier

$\Rightarrow$ CCF gives us a special form of A and B

that answers both these questions!

CCF

$$z[t+1] = \tilde{A} z[t] + \tilde{B} u[t]$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & & \cdots & 1 \\ a_0 & a_1 & & \cdots & a_{n-1} \end{bmatrix}}_{\tilde{A}} \vec{z}[t] + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\tilde{B}} u[t]$$

Characteristic polynomial:

$$\boxed{\chi_{\tilde{A}} = \lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda - a_0}$$

Note the $(-)$ signs!

Add feedback?

$$u = -\tilde{K} z$$

$$\tilde{A}_{cl} = \tilde{A} - \tilde{B}\tilde{K}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & & \cdots & 1 \\ a_0 - k_0 & & \cdots & & a_{n-1} - k_{n-1} \end{bmatrix}$$

$\chi_{\tilde{A}_{cl}}$ will still take the same form!

$$\chi_{\tilde{A}_{cl}} = \lambda^n - (a_{n-1} - k_{n-1})\lambda^{n-1} - \dots - (a_1 - k_1)\lambda - (a_0 - k_0)$$

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a) With this polynomial, we can place the eigenvalues anywhere!

Why?

In general, some char. poly. will look like:

$$\chi = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$

$\Rightarrow$ Matching coefficients of χ with $\chi_{\tilde{A}_{cl}}$, we see that exactly one k value controls each coefficient!

$\Rightarrow$ we can independently choose k for each coefficient and thus obtain any polynomial

$\Rightarrow$ any $\{\lambda\}$

b) How to choose k-values for large systems?

$$\begin{cases} c_{n-1} = -(a_{n-1} - k_{n-1}) \\ \vdots \\ c_1 = -(a_1 - k_1) \\ c_0 = -(a_0 - k_0) \end{cases}$$

But wasn't that for a system that just so happens to be in CCF?

$\Rightarrow$ Suppose there were a transformation T s.t.

$$\boxed{\begin{aligned} \vec{z} &= T\vec{x} \\ \tilde{A} &= TAT^{-1} \\ \tilde{B} &= TB \end{aligned}}$$

Then $\tilde{A}$ and A have the same eigenvalues!

$\tilde{A}$: controller basis

A: std basis

$\downarrow$

same characteristic polynomial

We could calculate the controller gains $\tilde{K}$ in the controller basis, and then transform back to the original basis to get the values of K we actually want!

std basis

controller basis

$$\begin{array}{ccc} A, B & \longrightarrow & \tilde{A}, \tilde{B} \quad (\text{CCF}) \\ \downarrow & & \downarrow \\ x(k) & \longleftarrow & \tilde{\ } \end{array}$$

Match x_{target} to $x(k)$

"direct" route

"indirect route"

easier to pick $\tilde{K}$ in controller basis

A_d $\quad K \longleftarrow K$

change of basis: use diagonalization as example

$$
\begin{tikzcd}
\vec{x} \arrow[rr, "A\vec{x}"] \arrow[d, shift right, "V^{-1}"'] & & \vec{y} = A\vec{x} \arrow[d, shift right, "V^{-1}"'] \\
\vec{x}_V \arrow[rr, "\Lambda = V^{-1}AV"'] \arrow[u, shift right, "V"'] & & \vec{y}_V = \Lambda \vec{x}_V \arrow[u, shift right, "V"']
\end{tikzcd}
$$

Returning to CCF:

To finish answering questions (a) and (b), we need to show that T exists for all controllable systems (so that we can transform to controller basis and solve the problem there)

(II.) "Derivation" and Procedure

From lecture:

$$\mathcal{C} = [\, B \;\; AB \;\; \cdots \;\; A^{n-1}B \,]$$

Let $q^T \equiv$ last row of $\mathcal{C}^{-1}$

Then $T = \begin{bmatrix} \vdash q^T \longrightarrow \\ \vdash q^T A \longrightarrow \end{bmatrix}$

$$\begin{bmatrix} \vdots \\ \vdash q^T A^{n-1} \dashv \end{bmatrix}$$

which ^has to exist because $\mathcal{C}^{-1}$ exists

(controllable $\Longleftrightarrow$ $\mathcal{C}$ is full rank $\Longleftrightarrow$ $\mathcal{C}^{-1}$ exists)

"Intuition":

$$TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \tilde{B}$$

If $q^T \equiv$ last row of $\mathcal{C}^{-1}$, then:

$$\mathcal{C}^{-1}\mathcal{C} = \begin{bmatrix} \vdots \\ \vdash q^T \dashv \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

$$= \begin{bmatrix} & \vdots & \\ q^TB & q^TAB & \cdots & q^TA^{n-1}B \end{bmatrix} = \begin{bmatrix} & \vdots & \\ 0 & \cdots & 0 \ 1 \end{bmatrix} = I$$

looks like $\tilde{B}^T$!

(Rest of proof finished off using Cayley-Hamilton to show $\tilde{A}T = TA$)

In practice will calculate T in this way:

$$\tilde{\mathcal{C}} \equiv [\tilde{B} \ \tilde{A}\tilde{B} \ \cdots \ \tilde{A}^{n-1}\tilde{B}]$$

$$= [TB \ \ TA\cancel{T^{-1}}TB \ \cdots \ TA^{n-1}\cancel{T^{-1}}TB]$$

$$= T[B \ AB \ \cdots A^{n-1}B]$$

$$= T\mathcal{C}$$

(Assumed controllable system $\Longrightarrow$ $\mathcal{C}$ is full rank)

$$\boxed{T = \tilde{\mathcal{C}}\mathcal{C}^{-1}}$$

And calculate K from $\tilde{K}$:

$$u = -\tilde{K}z = -Kx$$

$$\Longrightarrow \boxed{K = \tilde{K}T}$$

Procedure

① Find characteristic polynomial of A.

Since A and $\hat{A}$ have the same char. polynomial, set $\chi_A = \tilde{\chi}_{\tilde{A}}$:

$\Longrightarrow$ pattern match the coefficients to determine $a_0 \ldots a_{n-1}$ of $\tilde{A}$

② Calculate $T = \tilde{\mathcal{C}}\,\mathcal{C}^{-1}$

③ Place eigenvalues using $\tilde{K}$

④ Convert back to std basis to find $K = \tilde{K} T$

EX:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{controllable} \longrightarrow \text{CCF}$$

① $\chi_A = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = \underline{\lambda^2 - 2\lambda + 1}$

$$\chi_{\tilde{A}} = \det \begin{bmatrix} 0-\lambda & 1 \\ a_0 & a_1-\lambda \end{bmatrix} = \underline{\lambda^2 - a_1\lambda - a_0}$$

$$\Longrightarrow \quad -a_1 = -2 \Longrightarrow a_1 = 2$$

$$-a_0 = 1 \Longrightarrow a_0 = -1$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

② $\mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} B & AB \end{bmatrix}$

$$\tilde{\mathcal{C}} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} \end{bmatrix}$$

Find $\mathcal{C}^{-1} = \dfrac{1}{0\times 1 - 1\times 1} \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$$T = \tilde{\mathcal{C}}\,\mathcal{C}^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}$$

Check: $\tilde{B} = TB$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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(2) Eigenvalue Placement in CCF

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

a) controllable?

Already in CCF!
By definition, controllable

$\mathcal{C} = [B \;\; AB \;\; A^2B]$

b) stable?

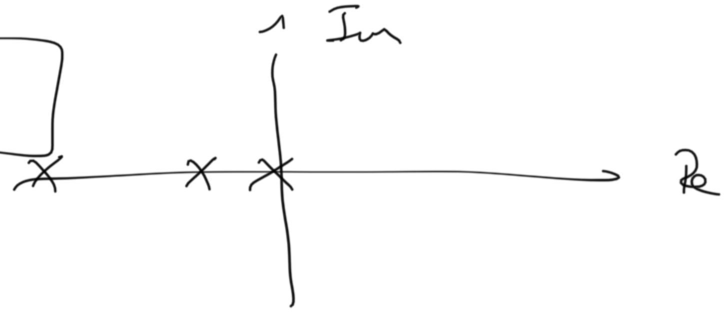
from CCF formula: $a_0 = 0$, $a_1 = -3$, $a_2 = -4$

$$\chi_A = \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$$

$$= \lambda^3 + 4\lambda^2 + 3\lambda = 0$$

$$\lambda(\lambda^2 + 4\lambda + 3) = \lambda(\lambda+1)(\lambda+3) = 0$$

$$\Longrightarrow \boxed{\lambda = 0, -1, -3}$$



$Re\{\lambda\} = 0$ $\Longrightarrow$ $\boxed{\text{marginally stable}}$

c) $u(t) = -K\vec{x}(t)$

Place λ at $\lambda = \underline{-1, -1}, -2$

Already in CCF; Don't need to use procedure from earlier (calculating T, $\mathcal{C}^{-1}$, etc.)

$$A - BK = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_0 & -k_1 - 3 & -k_2 - 4 \end{bmatrix} \\ \begin{array}{ccc} b_0 & \quad b_1 & \quad b_2 \end{array} \end{array}$$

$$\chi_{A_{cl}} = \lambda^3 - b_2\lambda^2 - b_1\lambda - b_0$$

$$= \lambda^3 - (-k_2 - 4)\lambda^2 - (-k_1 - 3)\lambda + k_0$$

Match to $\chi_{goal} = (\lambda+1)^2(\lambda+2)$

$$= \lambda^3 + 4\lambda^2 + 5\lambda + 2$$

$$\lambda^3 + 4\lambda^2 + 5\lambda + 2 = \lambda^3 - (-k_2 - 4)\lambda^2 - (-k_1 - 3)\lambda + k_0$$

$$\Rightarrow \quad k_2 + 4 = 4 \longrightarrow k_2 = 0$$
$$k_1 + 3 = 5 \longrightarrow k_1 = 2$$
$$k_0 = 2$$

$$\boxed{K = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}}$$

$$u = -Kx = \begin{bmatrix} -2 & -2 & 0 \end{bmatrix} x$$

3 CCF - Eigenvalue Placement

$$\vec{x}[t+1] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

Not in CCF!

a) Controllable?

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow \text{full rank}$$

$\boxed{\text{controllable!}}$

b) stable? (DT)

$$\det \begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} = (-1-\lambda)\big((-2-\lambda)(-1-\lambda) - 1\big)$$

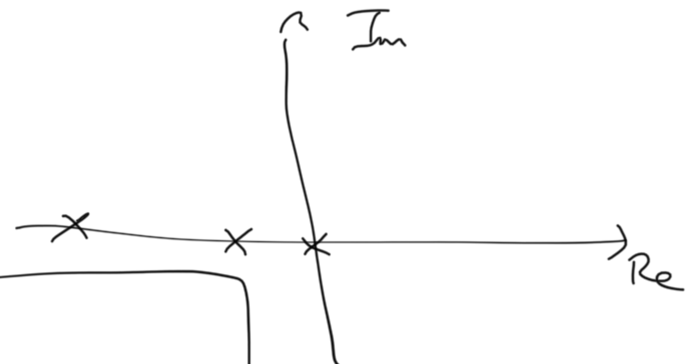
$$- 1\,(-1-\lambda - 0) + 0 = 0$$

$$= (-1-\lambda)\left[(\lambda+2)(\lambda+1) - 1\right] - (-1-\lambda)$$

$$= (-1-\lambda)\left[\lambda^2 + 3\lambda + \cancel{2 - 1 - 1}\right]$$

$$= (-1-\lambda)(\lambda^2 + 3\lambda) = (-1-\lambda)\,\lambda\,(\lambda+3)$$

$$\Longrightarrow \quad \lambda = -1,\ 0,\ -3$$



unstable!

have eigenvalues outside the unit circle

c) Calculate the transformation $T = \tilde{C}\, C^{-1}$

(1) $\chi_A = \lambda^3 + 4\lambda^2 + 3\lambda = \chi_{\tilde{A}}$

$\chi_{\tilde{A}} = \lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$

$$\Longrightarrow \quad a_2 = -4$$

$$a_1 = -3$$

$$a_0 = 0$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(2) $$\mathcal{E} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{\mathcal{E}} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}$$

From a calculator, $\mathcal{E}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$$T = \tilde{\mathcal{E}}\mathcal{E}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}}$$

d) Place eigenvals at $\lambda = 0, +\frac{1}{2}, -\frac{1}{2}$

$$u[t] = \tilde{K} z[t] = [\tilde{k}_0 \ \tilde{k}_1 \ \tilde{k}_2] z$$

$$\tilde{A}_{cl} = \tilde{A} + \tilde{B}\tilde{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tilde{k}_0 & \tilde{k}_1 - 3 & \tilde{k}_2 - 4 \end{bmatrix}$$

$$\tilde{\chi}_{A_{cl}} = \lambda^3 - (\tilde{k}_2 - 4)\lambda^2 - (\tilde{k}_1 - 3)\lambda - \tilde{k}_0$$

$$\chi_{goal} = \lambda\left(\lambda + \frac{1}{2}\right)\left(\lambda - \frac{1}{2}\right) = \lambda^3 - \frac{1}{4}\lambda$$

$$\Rightarrow \tilde{k}_2 - 4 = 0 \longrightarrow \tilde{k}_2 = 4$$

$$\tilde{k}_1 - 3 = \frac{1}{4} \longrightarrow \tilde{k}_1 = \frac{13}{4}$$

$$\tilde{k}_0 = 0$$

$$\boxed{\tilde{K} = \left[0 \quad \frac{13}{4} \quad 4\right]}$$

e) $K = \tilde{K} T$ (so $u[t] = K x[t]$)

$$= \left[0 \quad \frac{13}{4} \quad 4\right] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$= [\ \ 13 \quad \ \ -13 \ \ ]$$

$$\left[4 \quad \frac{\;}{4} - 12 \quad \frac{\;}{4} + 8\right]$$

$$\boxed{K = \left[4 \quad -\frac{35}{4} \quad \frac{19}{4}\right]}$$

Now you can do HW6, Q6 !