Discussion 7B

Discrete Fourier Transform (DFT)

* Motivation
* DFT Basis and Interpretation
* Calculating the DFT

I. Motivation for DFT

a) Why do we care about DFT
   (Fourier analysis in general) ?

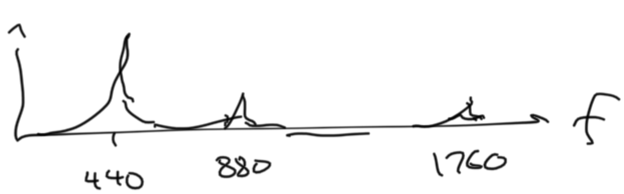
$\Rightarrow$ DFT (and Fourier analysis) allows us to look at "frequency components" of a signal!

Ex: Music

$\hookrightarrow$ What notes are in the song? What kind of overtones?

$\leftarrow$ A - 440 Hz
(tuning note)

Is it just a pure tone?

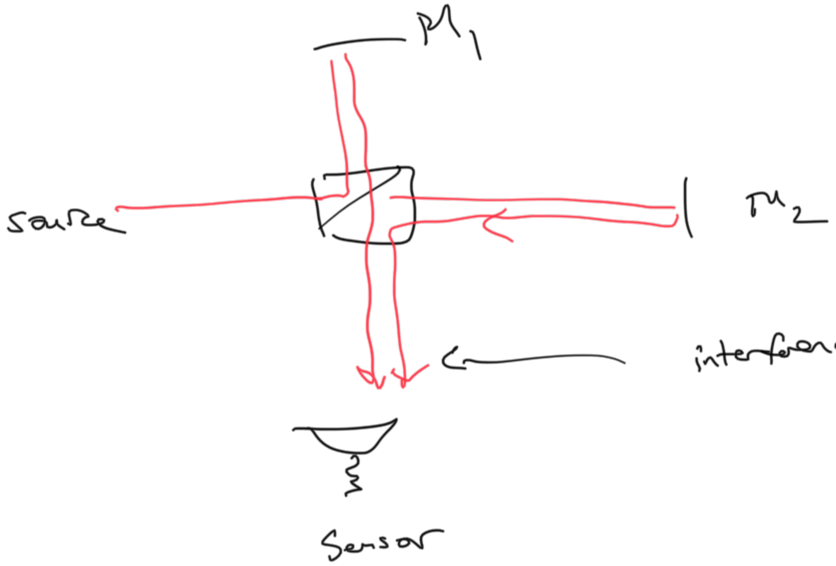


" " of instrument

$\Rightarrow$ determines timbre of mu...

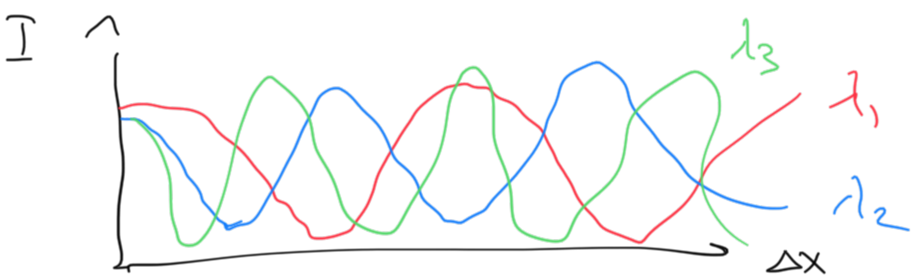
Ex: spectroscopy

What wavelengths / frequencies of light are present in my signal?



Michelson Interferometer

If I move M_2,

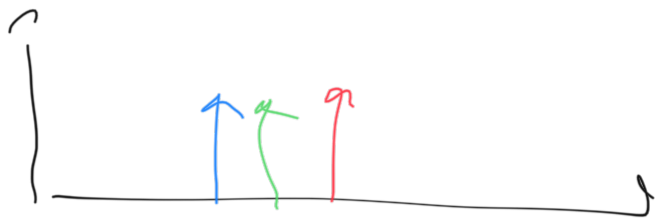


garbled sensor reading

$\Rightarrow$ can recover frequency components

Fourier Transform

using a Fourier transform



"Fourier Transform Spectroscopy"

(FTIR)

Ex: Image Compression

Your signal could be "busy" in time domain but sparse in frequency domain

↳ compression!

JPEG uses a related transform called DCT

(discrete cosine transform)

b) Why discrete?

↳ how computers do it

↳ FFT (fast Fourier transform)

$O(N \log N)$ instead of $O(N^2)$

Conclusion: signal processing is everywhere, and you should take EE120!

II. DFT Basis and Interpretation

length N signal:

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

DFT Basis:

$$\boxed{\text{kth basis vector} \quad \underbrace{\vec{u_k}}_{\text{"index"}}[n] = \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}kn} = \frac{1}{\sqrt{N}} \omega_N^{kn} \qquad \left(\omega_N = e^{i\frac{2\pi}{N}}\right)}$$

That is,

$$\vec{u_k} = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega_N^{k0} \\ \omega_N^{k1} \\ \vdots \\ \omega_N^{k(N-1)} \end{bmatrix} = \begin{bmatrix} u_k[0] \\ u_k[1] \\ \vdots \\ u_k[N-1] \end{bmatrix} \qquad \begin{matrix} k = 0 \ldots N-1 \\ n = 0 \ldots N-1 \end{matrix}$$

DFT Basis Matrix is given by:

$$U = \begin{bmatrix} \uparrow & & \uparrow \\ \vec{u_0} & \cdots & \vec{u_{N-1}} \\ \downarrow & & \downarrow \end{bmatrix}$$

* $\vec{u_j}$ are orthonormal
* U is unitary

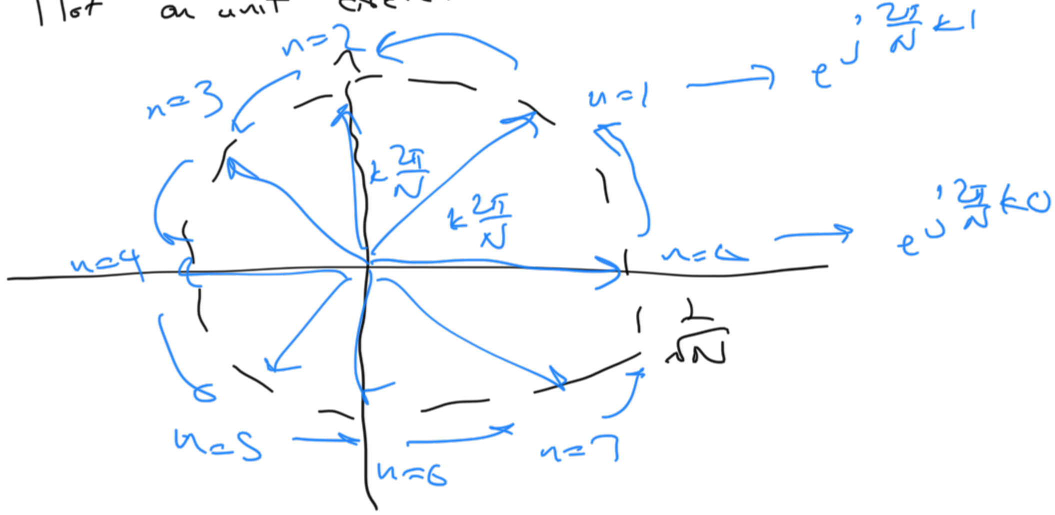
* $\| U z \| = \| z \|$

Stop here:

Let's take a closer look at what this all means

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega_N^{k0} \\ \vdots \\ \omega_N^{k(N-1)} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi}{N}k0} \\ e^{j\frac{2\pi}{N}k1} \\ \vdots \\ e^{j\frac{2\pi}{N}k(N-1)} \end{bmatrix}$$

Plot on unit circle:



Visually, moving around the unit circle
(some kind of cyclic behavior)

$$\vec{u}_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn}$$

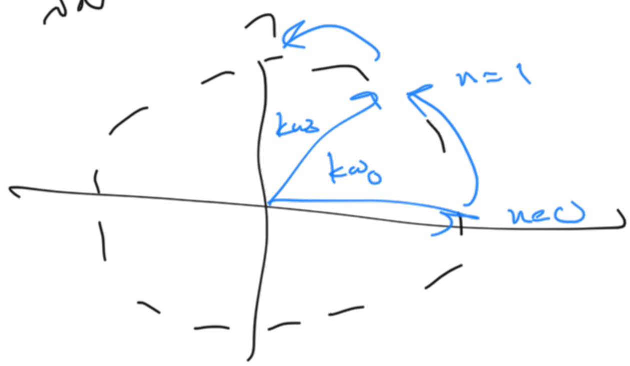
$f = \frac{1}{T}$

$\omega = \frac{2\pi}{T}$

Call $\frac{2\pi}{N} = \omega_0$ (fundamental frequency)

(eg: you have a N-periodic signal)

$$\Rightarrow \vec{u}_k[n] = \frac{1}{\sqrt{N}} e^{j(\omega_0 k) n}$$



- k determines how much $u_k[n]$ rotates with each increment of n
- k determines what harmonic of the fundamental frequency we are looking at!

$$\omega = k\omega_0$$

- n is the time index!

In summary,

Each DFT basis $\vec{u}_k$ represents a <u>pure frequency component</u> with frequency $\omega = k\omega_0 = k\frac{2\pi}{N}$

$$u_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn}$$

$\omega_0 = \frac{2\pi}{N}$: fundamental frequency

k: freq index

n: time index

Interpretation of the DFT Basis

Ex: $N=4$,

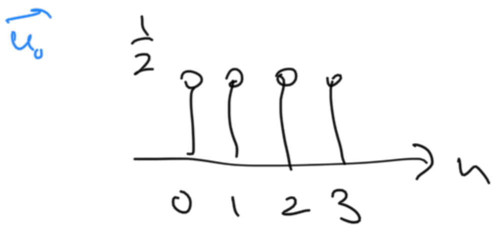
$$\vec{u}_k[n] = \frac{1}{\sqrt{4}} e^{j\frac{2\pi}{N}kn} = \frac{1}{2} e^{j\frac{2\pi}{4}kn} = \frac{1}{2} e^{j\frac{\pi}{2}kn}$$

$k=0$: $\vec{u}_0 \longrightarrow$ expect this to correspond to

$\omega = k\omega_0 = 0\,\omega_0 = 0$

$\longrightarrow$ DC component!

$$\vec{u}_0[n] = \frac{1}{2} e^{j\frac{\pi}{2}0n} = \frac{1}{2}$$



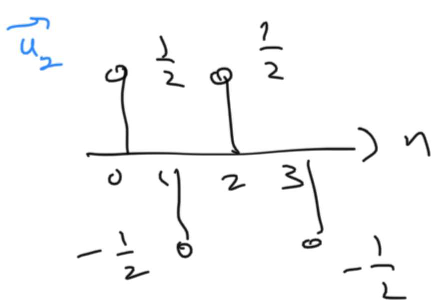
$\longleftarrow$ yep, DC!

$k=2$: $\vec{u}_2 \longrightarrow \omega = k\omega_0 = 2\frac{2\pi}{4} = \pi$

???

$$\vec{u}_2[n] = \frac{1}{2} e^{j\frac{\pi}{2}kn} = \frac{1}{2} e^{j\pi n} = \frac{1}{2}(-1)^n$$

$= \frac{1}{2}(e^{j\pi})^n = \frac{1}{2}(-1)^n$



$\longleftarrow$ high frequency signal

($\omega = \pi$ is highest digital frequency)

III. Calculating DFT

$\rightarrow$ be the time-domain signal, Let x [illegible]

$\vec{X}$ be freq domain representation (representation/coordinates in DFT basis)

From change of basis:

$$\vec{x} = U\vec{X} = \begin{bmatrix} | & & | \\ \vec{u}_0 & \cdots & \vec{u}_{N-1} \\ | & & | \end{bmatrix} \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\vec{x} = X[0]\vec{u}_0 + X[1]\vec{u}_1 + \dots + X[N-1]\vec{u}_{N-1}$$

$$= \sum_{k=0}^{N-1} X[k]\vec{u}_k$$

Time domain signal $\vec{x}$ is a linear combination of pure frequencies!

For some reason (by convention), we say that the Fourier matrix F is given by:

$$F = U^*$$

So $\vec{x} = U\vec{X} \implies \vec{x} = F^*\vec{X}$

$$\boxed{\vec{x} = F^*\vec{X} \quad \text{(Synthesis Equation)}}$$

$\vec{x} = F\ X$ [Synt... Eq...]

$\vec{X} = F\vec{x}$ [Analysis Equation]

$$F^{*} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{1\times 1} & \cdots & \omega_N^{(N-1)\times 1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{1\times(N-1)} & \cdots & \omega_N^{(N-1)\times(N-1)} \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{u}_0 & \cdots & \vec{u}_{N-1} \\ | & & | \end{bmatrix}$$

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(1) Roots of Unity

Solution to $z^N = 1$ $(z \in \mathbb{C})$

a) Show $z^N - 1 = (z-1)\sum_{k=0}^{N-1} z^k$

1 way: start from RHS

$$(z-1)\sum_{k=0}^{N-1} z^k = \sum_{k=0}^{N-1} z^{k+1} - z^k$$

$$= \sum_{k=0}^{N-1} z^{k+1} - \sum_{k=0}^{N-1} z^k$$

$$= z^1 + z^2 + \dots + z^N - (z^0 + z^1 + \dots + z^{N-1})$$

$$= z^N - z^0$$

$$= z^N - 1 \qquad \square$$

2nd way:

$$\sum_{k=0}^{N-1} z^k = z^0 + z^1 \dots + z^{N-1}$$

$$= 1 + z + z^2 + \dots + z^{N-1}$$

$$= \frac{1 - z^N}{1 - z} = \frac{z^N - 1}{z - 1}$$

Thus, $(z-1)\sum_{k=0}^{N-1} z^k = z^N - 1$ D

b) Show $\omega_k = e^{j\frac{2\pi}{N}k}$, $k \in \mathbb{Z}$ is a N^{th} root of unity

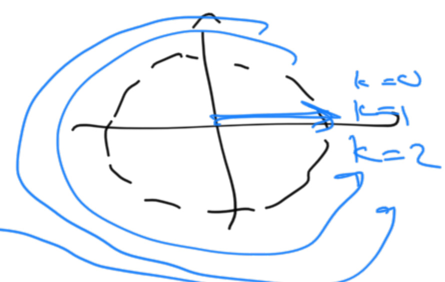
$$z^N - 1 = 0 \implies z^N = 1$$

Directly show:

$$\omega_k^N = \left(e^{j\frac{2\pi}{N}k}\right)^N = e^{j\frac{2\pi}{\cancel{N}}k\cancel{N}} = e^{j2\pi k}$$

$$= 1$$

So it satisfies $z^N = 1$



Subtlety!

$(e^{j2\pi})^k$ $k = \frac{1}{2}$? $\implies$ $\cancel{(e^{j2\pi})^{\frac{1}{2}} = 1}$?

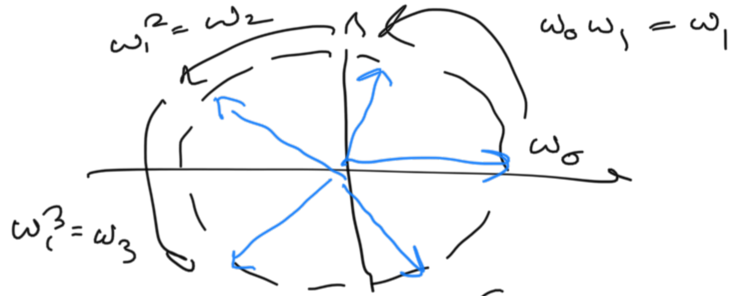
$$(e^{j2\pi})^{\frac{1}{2}} = e^{j2\pi\frac{1}{2}} = e^{j\pi} = -1$$

(A little subtle)

c) Fifth roots of unity

N^{th} roots of unity $\longrightarrow$ N distinct roots

$N = 5$: $e^{j\frac{2\pi}{5}0}$, $e^{j\frac{2\pi}{5}1}$, $e^{j\frac{2\pi}{5}2}$, $e^{j\frac{2\pi}{5}3}$, $e^{j\frac{2\pi}{5}4}$



d) $\omega_1 = e^{j\frac{2\pi}{5}}$ $\left(\omega_k = e^{j\frac{2\pi}{5}k} \right)$

What is ω_1^2? ω_1^5? ω_1^{42}?

$$\boxed{\omega_1^2 = \left(e^{j\frac{2\pi}{5}}\right)^2 = e^{j\frac{2\pi}{5}2} = \omega_2}$$

$$\boxed{\omega_1^3 = \left(e^{j\frac{2\pi}{5}}\right)^3 = e^{\left(j\frac{2\pi}{5}\right)3} = \omega_3}$$

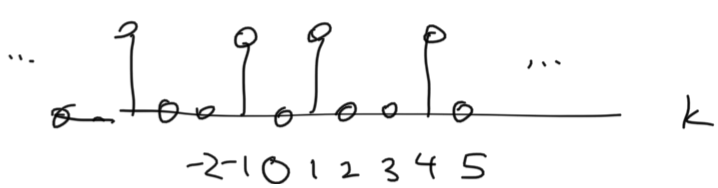
$$\omega_1^{42} = e^{j\frac{2\pi}{5}42} = e^{j\frac{2\pi}{5}(40+2)} = e^{j\frac{2\pi}{5}40} e^{j\frac{2\pi}{5}2}$$

$$= \cancel{e^{j2\pi(8)}} e^{j\frac{2\pi}{5}2} = \omega_2$$

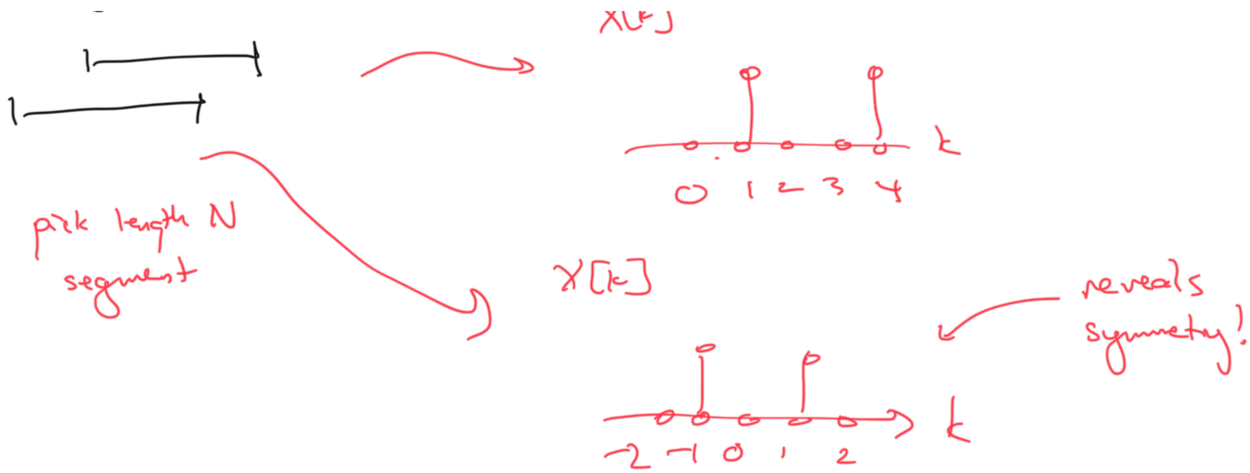
$$\boxed{\omega_1^{42} = \omega_2}$$

- Note that ω_N^{kn} is N-periodic in k!
- What matters is $k \bmod N$
- Often we will use negative values of k

Why? $\Longrightarrow$ center around $\underline{\underline{0}}$



$(N=5)$



e) $\overline{\omega_1}\,,\ \overline{\omega_{42}}$

$$\boxed{\begin{aligned}\overline{\omega_1} &= e^{-j\frac{2\pi}{5}} = \omega_{-1 \bmod 5} = \omega_4\\ \overline{\omega_{42}} &= \overline{\omega_2} = \omega_{-2 \bmod 5} = \omega_3\end{aligned}}$$

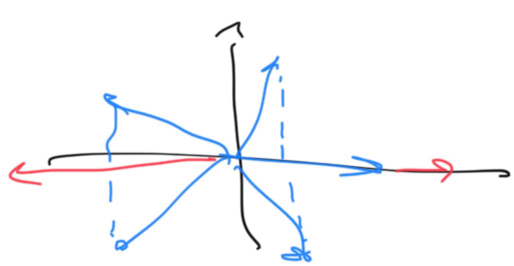
$$\omega_1 = e^{j\frac{2\pi}{N}1}$$

$$\overline{\omega_1} = e^{-j\frac{2\pi}{N}1} = \omega_{-1 \bmod N} = \omega_{N-1}$$

$$\overline{\omega_k} = \omega_{N-k} \qquad \text{(in general)}$$

f) $\displaystyle\sum_{k=0}^{N-1} \omega^k = ?$

(From HW1, you did graphically)



geometric argument that this should equal 0

$$\sum_{k=0}^{N-1} \omega^k = \frac{\omega^N - 1}{\omega - 1} \quad (\text{part (a)})$$

But if ω is a N^{th} root of unity,

by def: $\omega^N = 1$

Thus

$$\boxed{\sum_{k=0}^{N-1} \omega^k = \frac{1-1}{\omega - 1} = 0 \quad \text{if } \omega \neq 1}$$

edge case!

If $N=1$, $\omega = 1$

$$\Rightarrow \boxed{\sum_{k=0}^{N-1} \omega^k = \sum_{k=0}^{N-1} 1^k = N}$$

3.) DFT of Pure Sinusoids

Try to do without multiplying F!

a) $x(t) = \cos\left(\frac{2\pi}{3} t\right)$

$\downarrow$ sample 3 times, every 1 sec

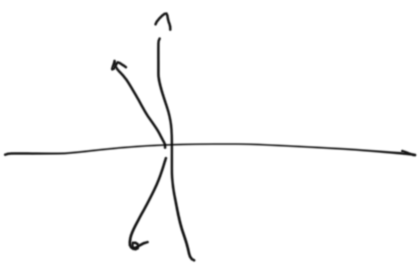
$$x[n] = \left[\cos\left(\frac{2\pi}{3} 0\right) \quad \cos\left(\frac{2\pi}{3} 1\right) \quad \cos\left(\frac{2\pi}{3} 2\right)\right]$$

Mechanically:

$0 \quad \omega_0 \quad 2\omega_0$

$$F^* = \begin{bmatrix} | & | & | \\ \vec{u}_0 & \vec{u}_1 & \vec{u}_2 \\ | & | & | \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} & & \\ 1 & e^{j\frac{2\pi}{3}} & e^{j(\frac{2\pi}{3}2)} \\ 1 & e^{j\frac{2\pi}{3}2} & e^{j(\frac{2\pi}{3}2)2} \end{bmatrix}$$

$$\vec{X} = F\vec{x} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-j\frac{2\pi}{3}2} \\ 1 & e^{-j\frac{2\pi}{3}2} & e^{-j\frac{2\pi}{3}4} \end{bmatrix} \begin{bmatrix} 1 \\ \cos(\frac{2\pi}{3}) = -\frac{1}{2} \\ \cos(\frac{4\pi}{3}) = -\frac{1}{2} \end{bmatrix}$$



$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 - \frac{e^{-j\frac{2\pi}{3}}}{2} - \frac{e^{-j\frac{2\pi}{3}2}}{2} \\ 1 - \frac{e^{-j\frac{2\pi}{3}2}}{2} - \frac{e^{-j\frac{2\pi}{3}4}}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 - \frac{1}{2}\left(-\frac{1}{2} \cancel{-j\frac{\sqrt{3}}{2}} + -\frac{1}{2} \cancel{+j\frac{\sqrt{3}}{2}}\right) \\ 1 - \frac{1}{2}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2} - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right) \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 - \frac{1}{2}(-1) \\ 1 - \frac{1}{2}(-1) \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ X[2] \end{bmatrix}$$

Not messy approach:

Remember that DFT decomposes $\vec{x}[n]$ into a linear combination of freq components

$$\ldots \; e^{j\frac{2\pi}{3}n} \; \ldots \; e^{-j\frac{2\pi}{3}n}$$

$\cos\left(\frac{2\pi}{3}n\right) = \frac{1}{2} e \quad + \quad \frac{1}{2} e$

$\vec{u}_k[n] = \frac{1}{\sqrt{3}} e^{j\frac{2\pi}{3}kn}$

$k=1$ $k=-1$

$\cos\left(\frac{2\pi}{3}n\right) = \frac{\sqrt{3}}{2}\vec{u}_1 + \frac{\sqrt{3}}{2}\vec{u}_{-1} = \frac{\sqrt{3}}{2}\vec{u}_1 + \frac{\sqrt{3}}{2}\vec{u}_2$

mod 3

$$\boxed{\begin{aligned} X[0] &= 0 \\ X[1] &= \frac{\sqrt{3}}{2} \\ X[2] &= \frac{\sqrt{3}}{2} \end{aligned}}$$

$\vec{x} = X[0]\vec{u}_0 + X[1]\vec{u}_1 + X[2]\vec{u}_2$

$= 0\,\vec{u}_0 + \frac{\sqrt{3}}{2}\vec{u}_1 + \frac{\sqrt{3}}{2}\vec{u}_2$

b) $N=6$

$x[n] = \left[\cos\left(\frac{2\pi}{3}0\right) \quad \dots \quad \cos\left(\frac{2\pi}{3}5\right)\right]$

Since $N=6$, $U = [\vec{u}_0\ \vec{u}_1\ \dots\ \vec{u}_5]$

$\vec{u}_k[n] = \frac{1}{\sqrt{6}} e^{j\frac{2\pi}{6}kn}$

$x[n] = \cos\left(\frac{2\pi}{3}n\right) = \frac{1}{2}e^{j\frac{2\pi}{3}n} + \frac{1}{2}e^{-j\frac{2\pi}{3}n}$

$= \frac{1}{2}e^{j\frac{4\pi}{6}n} + \frac{1}{2}e^{-j\frac{4\pi}{6}n}$

$k=2$ $k=-2$

$\left(2\,\frac{2\pi}{6} = \frac{2\pi}{3}\right) \qquad \left(-2\,\frac{2\pi}{6} = -\frac{2\pi}{3}\right)$

$$= \frac{1}{2}\sqrt{6}\,\vec{u_2} + \frac{1}{2}\sqrt{6}\,\vec{u_{-2}}$$

mod 6

$$= \underbrace{\frac{1}{2}\sqrt{6}}_{X[2]}\,\vec{u_2} + \underbrace{\frac{1}{2}\sqrt{6}}_{X[4]}\,\vec{u_4}$$

$$\boxed{\begin{aligned} &X[2] = X[4] = \frac{\sqrt{6}}{2} \\ &X[0] = X[1] = X[3] = X[5] = 0 \end{aligned}}$$

$$\vec{X} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{6}/2 \\ 0 \\ \sqrt{6}/2 \\ 0 \end{bmatrix}$$

Why are $X[-2]$ and $X[4]$ equal?

coefficients of

$e^{-j\frac{2\pi}{6}2n}$ vs $e^{j\frac{2\pi}{6}4n}$

$$\underbrace{\left(e^{j2\pi n}\right)}_{1} e^{-j\frac{2\pi}{6}2n} = e^{j\left(2\pi n - \frac{2\pi}{6}2n\right)}$$

$$= e^{j\frac{2\pi}{6}(6n-2n)}$$

$$= e^{j\frac{2\pi}{6}4n}$$

c) General case:

$$x(t) = \cos\left(\frac{2\pi}{N}mt\right),$$

$m = 0 \ldots N-1$

If $x[n]$ is length N,

$$x[n] = \left[\cos\left(\frac{2\pi}{N} m(0)\right) \quad \dots \quad \cos\left(\frac{2\pi}{N} m(N-1)\right) \right]$$

$$x[n] = \cos\left(\frac{2\pi}{N} mn\right) = \frac{1}{2} e^{j\frac{2\pi}{N}mn} + \frac{1}{2} e^{-j\frac{2\pi}{N}mn}$$

$$= \frac{\sqrt{N}}{2} \vec{u}_m + \frac{\sqrt{N}}{2} \vec{u}_{-m}$$

$$= \frac{\sqrt{N}}{2} \vec{u}_m + \frac{\sqrt{N}}{2} \vec{u}_{N-m}$$

$$\boxed{\begin{aligned} X[m] &= \frac{\sqrt{N}}{2} \\ X[N-m] &= \frac{\sqrt{N}}{2} \\ &\text{all else is } 0 \end{aligned}}$$

for $m \neq 0$

Special case: $x[n] = 1 \quad (m=0)$

$$\Rightarrow x[n] = 1 = X[0] \left.\frac{e^{j\frac{2\pi}{N}0n}}{\sqrt{N}}\right|_{k=0}$$

$$1 = X[0] \frac{1}{\sqrt{N}}$$

$$\Rightarrow \boxed{\begin{aligned} X[0] &= \sqrt{N} \\ &\text{all else is } 0 \end{aligned}}$$

for $m=0$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\cos\left(\frac{2\pi}{N}mn\right) = \frac{e^{j\left(\frac{2\pi}{N}mn\right)} + e^{-j\left(\frac{2\pi}{N}mn\right)}}{2}$$

"x"