Discussion 7C

DFT II

* Recap of DFT
* Example DFTs: sinusoid, constant signal, boxcar

(I.) Recap of DFT

decomposes a time-domain signal as a linear combination of pure frequency components

Call these frequency components DFT Basis:

For a length N signal:

$$\vec{x} = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

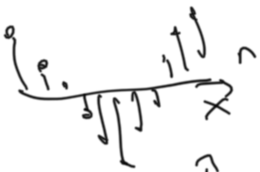
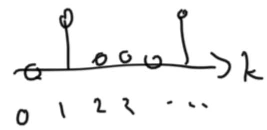
$$\boxed{\vec{u}_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn} = \frac{1}{\sqrt{N}} \omega_N^{kn}}$$

fundamental frequency $\omega_0 = \frac{2\pi}{N}$

k: frequency index

n: time index

$$U = \begin{bmatrix} u_0 & \cdots & u_{N-1} \end{bmatrix}$$

 $\vec{x} = U \vec{X}$ 

std basis (time domain)

DFT basis (freq domains)

$$\vec{x} = \sum_{k=0}^{N-1} X[k] \vec{u}_k$$

Or for each time index,

$$\boxed{x[n] = \sum_{k=0}^{N-1} X[k] u_k[n]}$$

IDFT, summation form

Define Fourier matrix: $\overline{F} \equiv U$

$F \equiv \overline{U}$

$\Rightarrow$ $\vec{x} = \overline{F} \vec{X}$ (synthesis equation)

$\vec{X} = F \vec{x}$ (analysis equation)

$$\overline{F} = U = \begin{array}{cccc} k=0 & k=1 & \cdots & k=N-1 \\ \left[\begin{array}{c} \omega_N^{0\cdot 0} \\ \omega_N^{0\cdot 1} \\ \vdots \end{array}\right. & \begin{array}{c} \omega_N^{1\cdot 0} \\ \omega_N^{1\cdot 1} \\ \vdots \end{array} & & \left.\begin{array}{c} \omega_N^{N-1\cdot 0} \\ \vdots \\ \vdots \end{array}\right] \end{array}$$

symmetric?

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ \omega_N^{0\cdot(N-1)} & \omega_N^{1\cdot(N-1)} & \cdots & \omega_N^{(N-1)\cdot(N-1)} \end{bmatrix}$$

$\omega_N = e^{j\frac{2\pi}{N}} \longleftarrow$ complex

In general, $\overline{F} \neq F$

(Some) Properties of the DFT

- $\vec{u}_k$ are orthonormal
  - $\hookrightarrow$ F is unitary $(F^* = \overline{F^T} = F^{-1})$
  - $\hookrightarrow$ $\|\vec{X}\|^2 = \|F\vec{x}\|^2 = \|\vec{x}\|^2$ (length preserving)
  - $\therefore \boxed{\|\vec{X}\|^2 = \|\vec{x}\|^2}$ (Parseval's Theorem)

  "energy conservation"

- F is symmetric
  - $\hookrightarrow$ $F^T = F$
  - $\hookrightarrow$ $F^* = \overline{F^T} = \overline{F} = F^{-1}$

  $\overline{F} \Leftarrow F^*$

  Then $\boxed{\overline{F} = F^{-1}}$

- DFT is linear

complex conjugate, NOT adjoint (complex conjugate + transpose)

- Conjugate symmetry:

  if $\vec{x} \in \mathbb{R}^N$ (real-valued signal) time-domain signal

$$\underbrace{X[k]}_{\text{DFT coefficients}} = \overline{X[-k]} = \overline{X[N-k]}$$

follows from N-periodicity of ω_N^k

- Duality:

From yesterday's lecture,

If $\vec{x}, \vec{X}$ are real:

$$\overline{F \vec{x}} = \overline{\vec{X}} \quad \longleftarrow \text{complex conjugate, } \underline{\text{NOT}} \text{ adjoint}$$

DFT of $\vec{x}[n]$ (time domain) is $\vec{X}[k]$ (freq. domain)

$$\overline{F}\,\overline{\vec{x}} = \overline{\vec{X}}$$

But $\overline{\vec{x}} = \vec{x}$, $\overline{\vec{X}} = \vec{X}$, so:

$$\overline{F} \vec{x} = \vec{X}$$

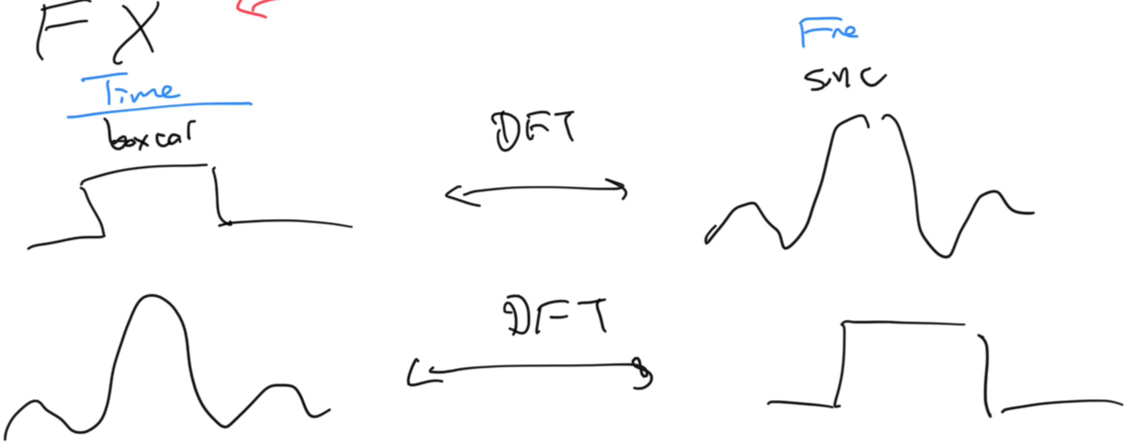
Note: This does $\underline{\underline{\text{NOT}}}$ mean $F = \overline{F}$! That would only be true if $F\vec{x} = \overline{F}\vec{x}$ $\forall \vec{x}$, but here we are showing for specific $\vec{x}$.

Since $\overline{F} = F^* = F^{-1}$:

$$\vec{x} = F \vec{X}$$

DFT of $\vec{X}[n]$ (time domain) is $\vec{x}[k]$ (freq. domain)

"duality":



(I) F, also DFTs

(Recap of Q2 DFS → B)

a) Cosine

$$x[n] = \cos\left(\frac{2\pi}{3} n\right) \quad , \quad n=0,1,2 \quad (N=3)$$

F matrix: tedious!

$$x[n] = \frac{1}{2} e^{j\frac{2\pi}{3}n} + \frac{1}{2} e^{-j\frac{2\pi}{3}n} \qquad e^{j2\pi n} = e^{j\frac{2\pi}{3}2n}$$

k

$$= X[0]\vec{u}_0[n] + X[1]\vec{u}_1[n] + X[2]\vec{u}_2[n]$$

$$= X[0]\frac{1}{\sqrt{3}}e^{j\frac{2\pi}{N}0n} + X[1]\frac{1}{\sqrt{3}}e^{j\frac{2\pi}{N}1n} + X[2]\frac{1}{\sqrt{3}}e^{j\frac{2\pi}{N}2n}$$

match coefficients

$$= X[0] + X[1]\frac{1}{\sqrt{3}}e^{j\frac{2\pi}{3}n} + X[2]\frac{1}{\sqrt{3}}e^{j\frac{2\pi}{3}2n}$$

$$= 0\,\vec{u}_0 + \frac{\sqrt{3}}{2}\underbrace{\frac{1}{\sqrt{3}}e^{j\frac{2\pi}{3}n}}_{\vec{u}_1} + \frac{\sqrt{3}}{2}\underbrace{\frac{1}{\sqrt{3}}e^{j\frac{2\pi}{3}n}}_{\vec{u}_{-1} = \vec{u}_2}$$

So $$\boxed{\vec{X} = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}}$$

General case:

$$\cos\left(\frac{2\pi}{N}mn\right) \quad n=0,1,\ldots,N-1$$

$x[n] = \cos\left(\frac{\;}{N} \right)$, ... (length N)

$$x[n] = \frac{1}{2} e^{j\frac{2\pi}{N} m n} + \frac{1}{2} e^{-j\frac{2\pi}{N} m n}$$

$k = m$

$$e^{-j\frac{2\pi}{N} mn} e^{j\frac{2\pi}{N} Nn} = e^{j\frac{2\pi}{N}(N-m)n} \Longrightarrow k = N - m$$

$$= \frac{\sqrt{N}}{2} \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N} mn} + \frac{\sqrt{N}}{2} \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}(N-m)n}$$

$$\boxed{X[k] = \begin{cases} \frac{\sqrt{N}}{2} & k = m,\ N-m \\ 0 & \text{else} \end{cases}}$$

b) constant function

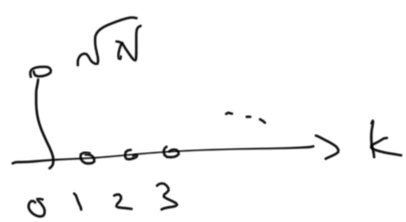
length N

$$x[n] = 1 = X[0]\, \vec{u}_0$$

$$\vec{u}_0 = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

So $X[0] = \sqrt{N}$, 0 else

$$\boxed{X[k] = \begin{cases} \sqrt{N} & k = 0 \\ 0 & \text{else} \end{cases}}$$



c) boxcar (rectangular pulse)

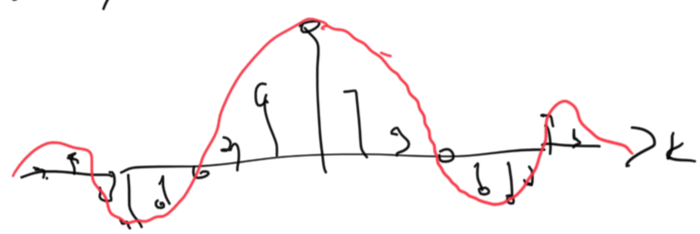
$$x[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{else} \end{cases}$$

Answer (after a lot of math)

$$X[k] = \frac{1}{\sqrt{N}} \frac{\sin\left[\frac{\pi}{N}(2M+1)k\right]}{\sin\left(\frac{\pi}{N}k\right)}$$

"sampled periodic sinc"

(Dirichlet function)



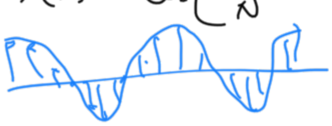
For all intents and purposes,

$$\text{boxcar} \xleftrightarrow{\text{DFT}} \text{sinc}$$

$$\text{sinc} \xleftrightarrow{\text{DFT}} \text{boxcar}$$

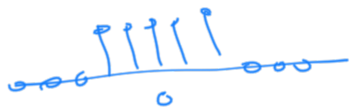
Summary of DFT pairs so far:

$$x[n] = \cos\left(\frac{2\pi}{N}mn\right) \xrightarrow{\text{DFT}} X[k] = \begin{cases} \frac{\sqrt{N}}{2} & k = m, N-m \\ 0 & \text{else} \end{cases}$$



$$x[n] = 1 \xrightarrow{\text{DFT}} X[k] = \delta[k] = \begin{cases} \sqrt{N} & k = 0 \\ 0 & \text{else} \end{cases}$$

$x[n]$ = boxcar $\xrightarrow{\text{DFT}}$ $X[k]$ = sinc



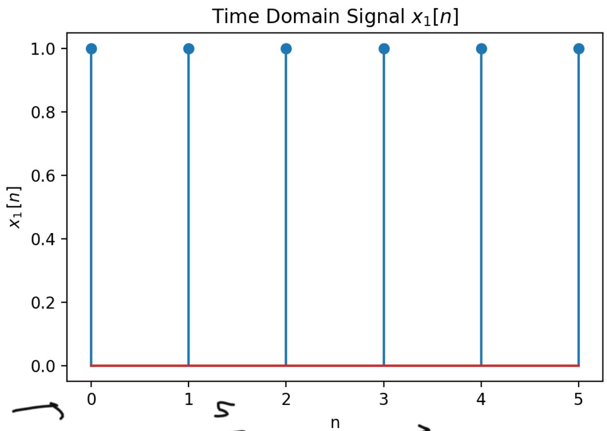
$x[n]$ = sinc $\xrightarrow{\text{DFT}}$ $X[k]$ = box car

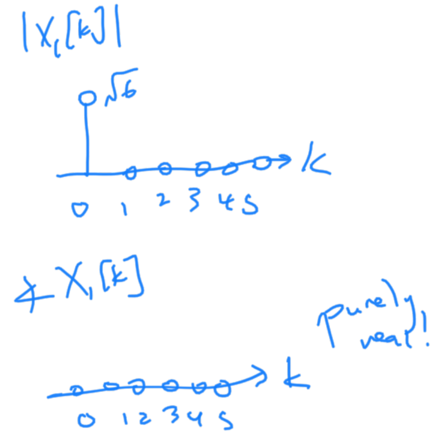
Exercise for the reader:
After doing Dis 7C, add to this list!!!

1 DFT

Consider the following length 6 signals. Compute its DFT coefficients $X[k]$. Then plot its magnitude $|X[k]|$ and phase $\angle X[k]$.

a) $x_1[n] = u[n] = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$.





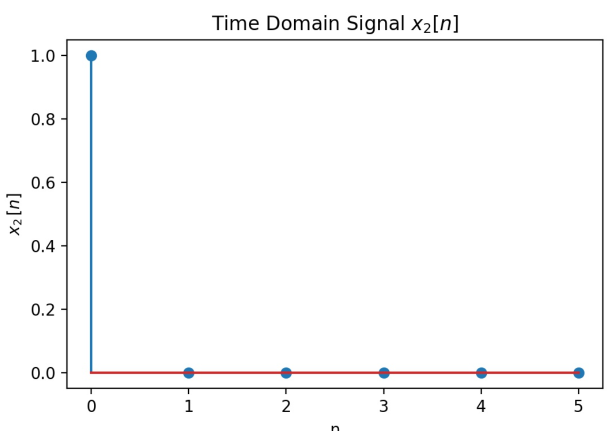
$$\vec{x}_1 = F^* \vec{X}_1 = \sum_{k=0}^{5} X_1[k]\,\vec{u}_k$$

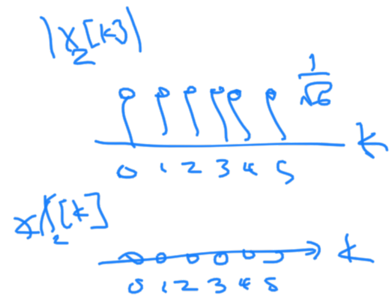
But $\vec{u}_0 = \frac{1}{\sqrt6}\begin{bmatrix}1 & 1 & 1 & \cdots & 1 & 1 & 1\end{bmatrix}^T$, so $\vec{x}_1 = X_1[0]\,\vec{u}_0$

$$\implies X_1[0] = \sqrt6$$

$$\implies \boxed{X_1[k] = \begin{cases}\sqrt6 & k=0\\ 0 & \text{else}\end{cases}}$$

b) $x_2[n] = \delta[n] = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$.



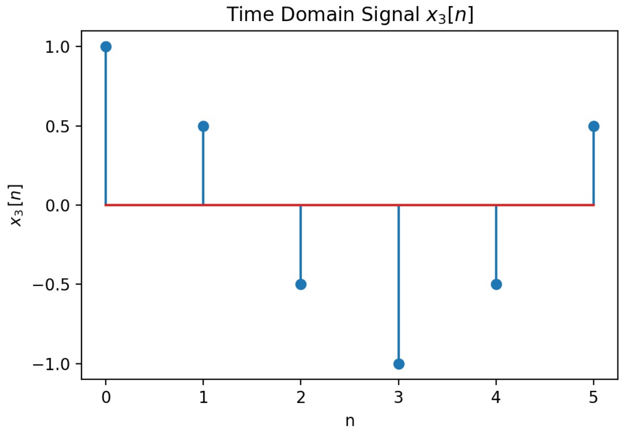


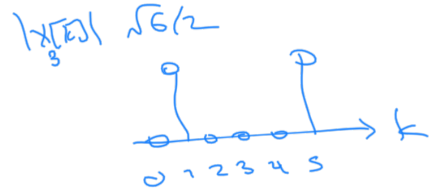
$$\vec{X}_2 = F\,\vec{x}_2 = \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} = \vec{u}_0 = \frac{1}{\sqrt6}\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}$$

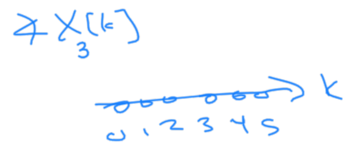
also purely real!

Thus, $\boxed{X_2[k] = 1/\sqrt6}$

c) $x_3[n] = \cos\left(\frac{2\pi}{6}n\right)$ for $n = 0, 1, \ldots, 5$.







one oscillation → fundamental frequency → $m = 1$

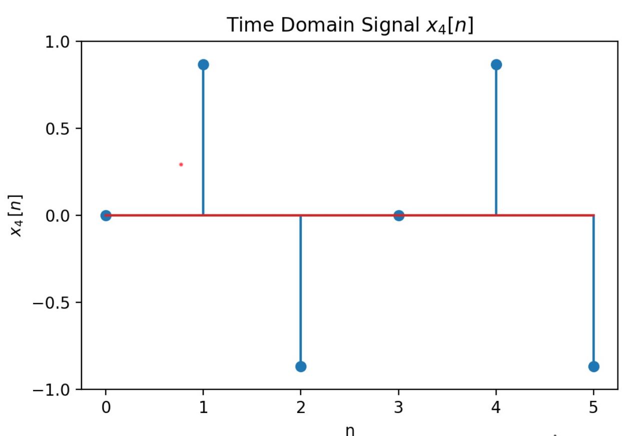
$\cos\left(\frac{2\pi}{N} mn\right) \longrightarrow X_3[k] = \begin{cases} \frac{\sqrt{N}}{2} & k = m, N-m \\ 0 & \text{else} \end{cases}$ becomes $\boxed{X_3[k] = \begin{cases} \frac{\sqrt{6}}{2} & k = 1, 5 \\ 0 & \text{else} \end{cases}}$

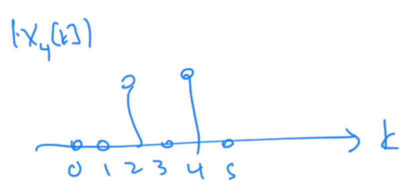
In math:

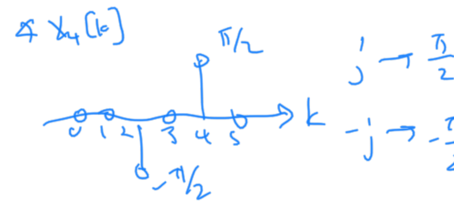
$\cos\left(\frac{2\pi}{6} n\right) = \frac{1}{2}\sqrt{6}\frac{1}{\sqrt{6}} e^{j\frac{2\pi}{6}n} + \frac{1}{2}\sqrt{6}\frac{1}{\sqrt{6}} e^{-j\frac{2\pi}{6}n} = \frac{\sqrt{6}}{2}\left(\vec{u}_1 + \vec{u}_{-1}\right)$

$-1 \bmod 6 = 5$

d) $x_4[n] = \sin\left(\frac{4\pi}{6}n\right)$ for $n = 0, 1, \ldots, 5$.







2 periods, expect nonzero at $k = 2, 4$ (second harmonic)

In math:

$\sin\left(\frac{4\pi}{6} n\right) = \frac{1}{2j}\left(\sqrt{6}\frac{1}{\sqrt{6}} e^{j\frac{4\pi}{6}n} - \sqrt{6}\frac{1}{\sqrt{6}} e^{-j\frac{4\pi}{6}n}\right)$

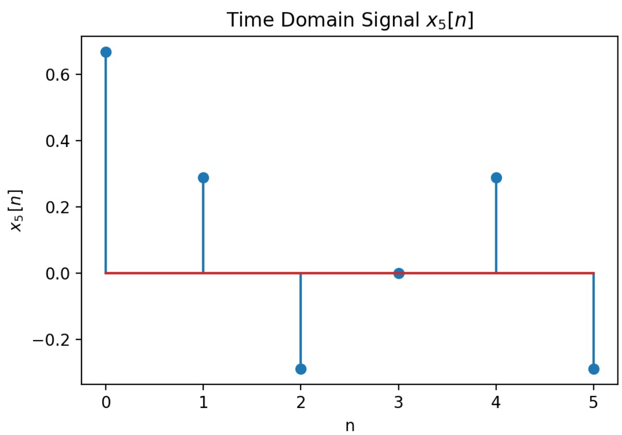
$\frac{4\pi}{6} = \frac{2\pi}{6} m \implies m = 2$

$= \frac{\sqrt{6}}{2j}\vec{u}_2 - \frac{\sqrt{6}}{2j}\vec{u}_{-2}$

$= -j\frac{\sqrt{6}}{2}\vec{u}_2 + \frac{\sqrt{6}}{2}j\,\vec{u}_4$

$\boxed{X_4[k] = \begin{cases} -j\frac{\sqrt{6}}{2} & k = 2 \\ j\frac{\sqrt{6}}{2} & k = 4 \\ 0 & \text{else} \end{cases}}$

e) $x_5[n] = \frac{2}{3}x_2[n] + \frac{1}{3}x_4[n]$



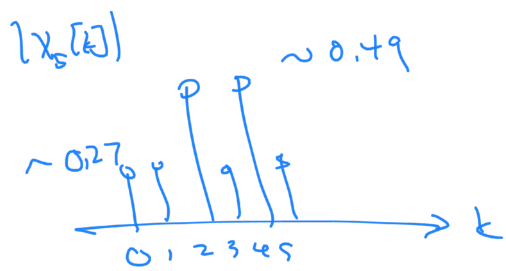
DFT is linear!

Add $\frac{2}{3} X_2[k] + \frac{1}{3} X_4[k]$ to get $X_5[k]$

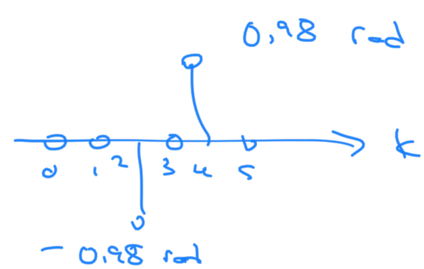
$$\Rightarrow X_5[k] = \frac{2}{3}\frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} + \frac{1}{3}\frac{\sqrt{6}}{2j}\begin{bmatrix}0\\0\\1\\0\\-1\\0\end{bmatrix} = \frac{2}{3}\frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix} + \frac{-j}{\sqrt{6}}\begin{bmatrix}0\\0\\1\\0\\-1\\0\end{bmatrix}$$

$$\boxed{X_5[k] = \frac{1}{\sqrt{6}}\begin{bmatrix}2/3\\2/3\\2/3 - j\\2/3\\2/3 + j\\2/3\end{bmatrix}}$$

$$|X_5[k]| = \frac{1}{\sqrt{6}}\begin{bmatrix}2/3\\2/3\\\sqrt{4/9+1}\\2/3\\\sqrt{4/9+1}\\2/3\end{bmatrix}$$



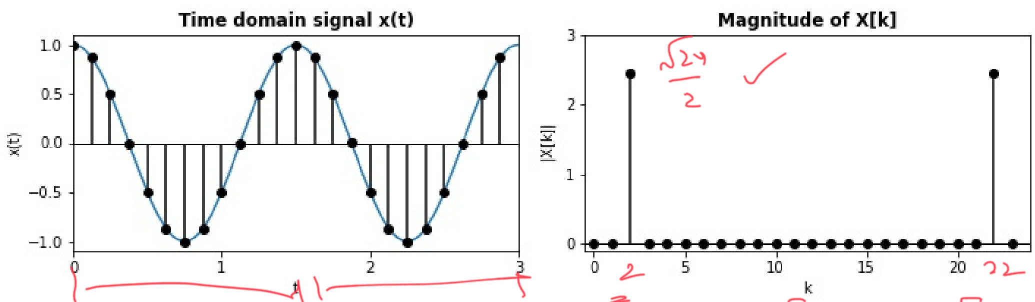
$$\measuredangle X_5[k] = \begin{bmatrix}0\\0\\\text{atan2}(-1, 2/3)\\0\\\text{atan2}(1, 2/3)\\0\end{bmatrix}$$



2 DFT Sampling Matching

Select the correct answer from the multiple choice options provided and give some justification.

a) A sampled time domain signal and its DFT coefficients are given below:

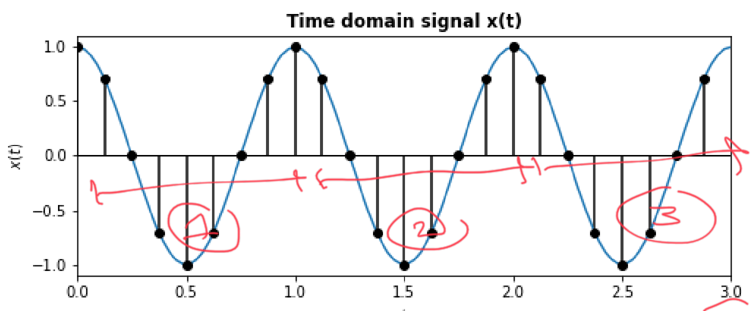


$X[2] = X[24-2] = \frac{\sqrt{24}}{2}$ ✓

Now given the following time domain signal, which of the options below shows the correct DFT coefficient magnitudes?

DFT: cosine $\overset{DFT}{\longleftrightarrow}$ two lollipops

$\cos\left(\frac{2\pi}{N} mn\right) \overset{DFT}{\longleftrightarrow} X[k] = \begin{cases} \frac{\sqrt{N}}{2} & k = m, N-m \\ 0 & \text{else} \end{cases}$



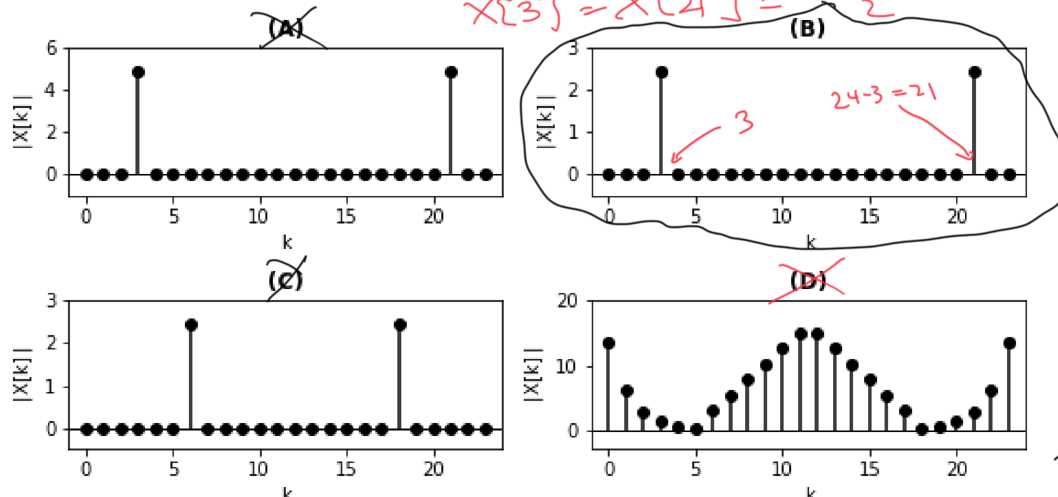
24 samples $\Downarrow$ $N = 24$

$\Downarrow$ $X[n] = X[N-m] = \frac{\sqrt{24}}{2} \approx 2.5$

$m = 3$

$X[3] = X[21] = \frac{\sqrt{24}}{2}$

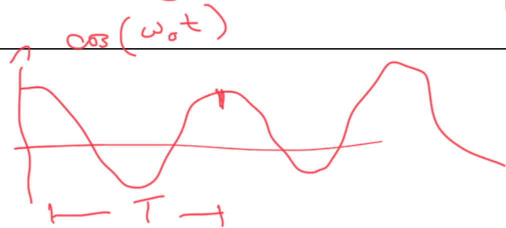
— m tells us what harmonic (multiple of ω_0) we are looking at!



To figure out m, count the number of complete cycles! $\longleftarrow$ If $\omega_0 = \frac{2\pi}{N}$: $\cos\left(\frac{2\pi}{N} n\right) = \cos(\omega_0 n)$

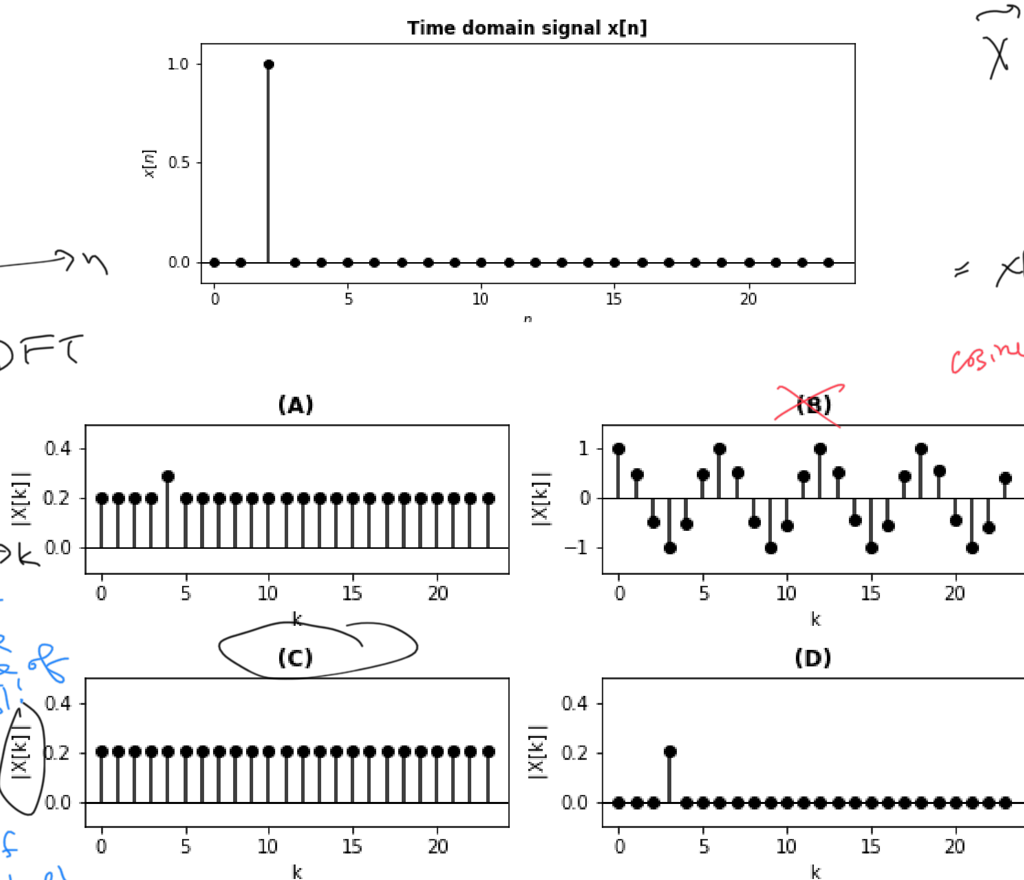
The period is N $\Downarrow$ one oscillation/complete cycle!

$\omega_0 = \frac{2\pi}{T}$ $\cos(\omega_0 t)$



Ex: If oscillating at $2\times$ the fundamental freq ω_0, make 2 complete cycles

b) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?



Mathematically.

$\vec{X} = F\vec{x}$

$= \begin{bmatrix} \bar{u}_0^T & \bar{u}_1^T & \cdots & \bar{u}_{N-1}^T \end{bmatrix} \vec{x}$

$= x[0]\bar{u}_0 + \dots + x[N-1]\bar{u}_{N-1}$

DFT

$|X[k]|$

cosine, should correspond to 2 impulses/lollipops

$\vec{X} = \begin{bmatrix} \bar{u}_0 & \cdots & \bar{u}_{N-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$

$= \bar{u}_2$

$= \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$

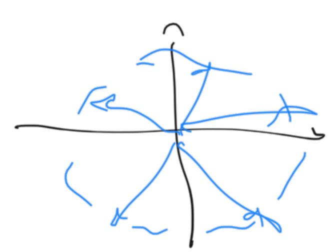
Note that this is the magnitude of each $|X[k]|$. The phase changes if impulse delayed different amounts.

$k = 2$

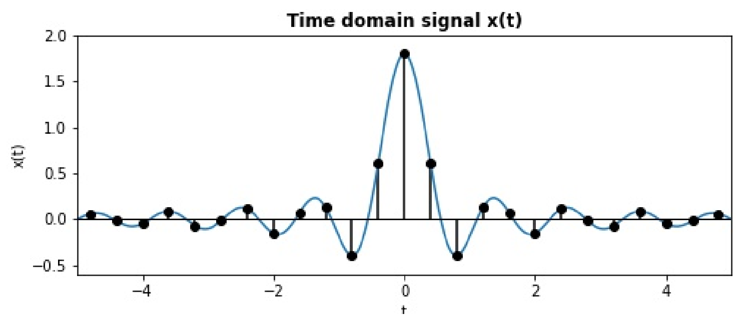
$$\vec{u}_2 = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega_N^{2(0)} \\ \omega_N^{2(1)} \\ \vdots \\ \omega_N^{2(N-1)} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi}{N}2(0)} \\ e^{j\frac{2\pi}{N}2(1)} \\ \vdots \\ e^{j\frac{2\pi}{N}2(N-1)} \end{bmatrix}$$

Magnitude of each component

$$\begin{bmatrix} |X[0]| \\ |X[1]| \\ \vdots \\ |X[N-1]| \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$



c) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?

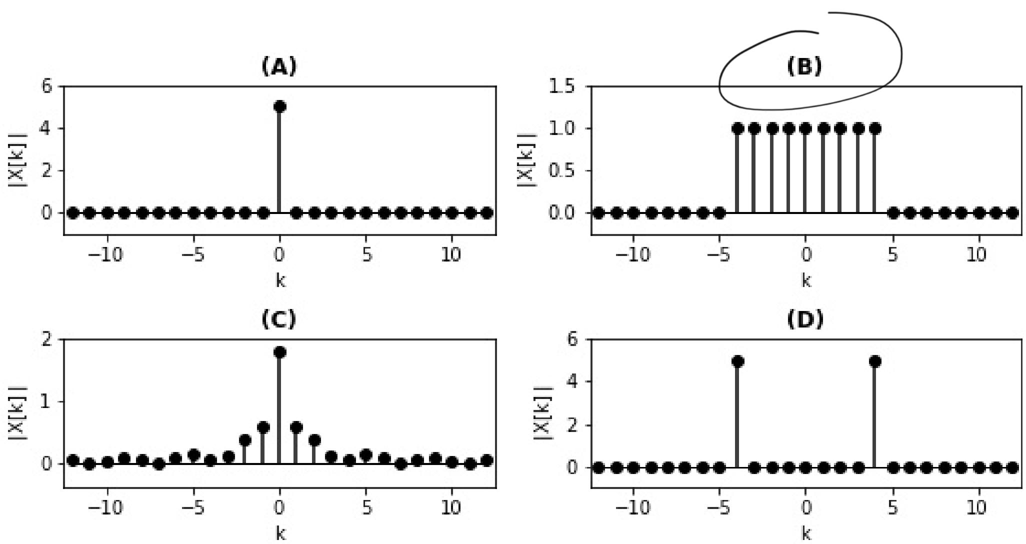


boxcar

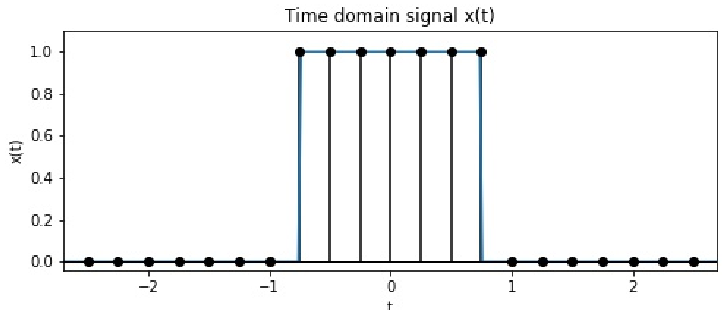
DFT

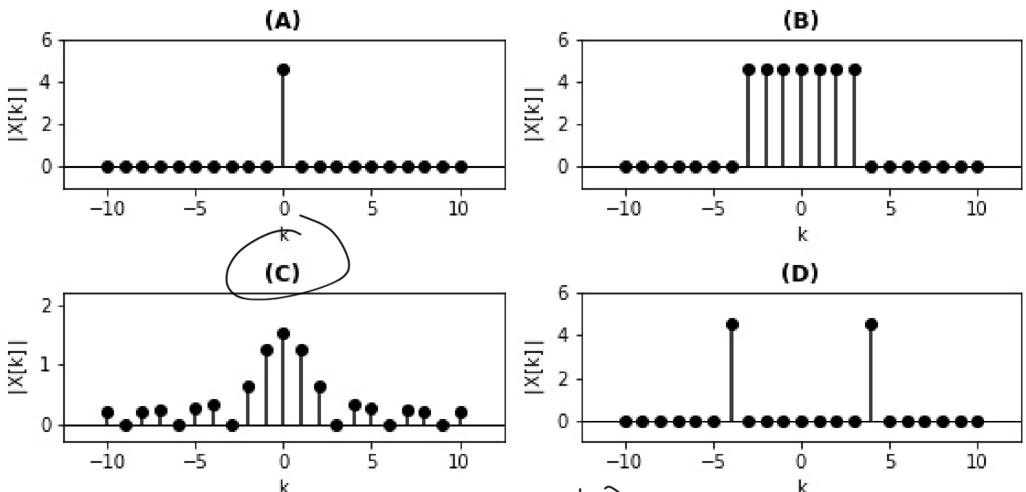
sinc

(from lecture)



d) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?





(Note the absolute value!)

sinc

$\updownarrow$ DFT

boxcal

(from lecture, duality)