Discussion 7D

- LTI Systems
  - Definition
  - Checking for LTI Example
- Convolution and Impulse Responses

(0.) Some notes on duality

If $\vec{x}, \vec{X}$ are purely real (so $\vec{x} = \overline{\vec{x}}$, $\vec{X} = \overline{\vec{X}}$)

$$F\vec{x} = \vec{X} \quad \text{(analysis eqn)} \qquad (1)$$

$$\rightarrow \overline{F}\,\overline{\vec{x}} = \overline{\vec{X}} \implies \overline{F}\vec{x} = \vec{X}$$

But since F is symmetric ($F^T = F$),

then $F^* = \overline{F}^T = \overline{F}$

So $\overline{F}\vec{x} = \vec{X} \longrightarrow F^*\vec{x} = \vec{X}$

$$FF^*\vec{x} = F\vec{X}$$

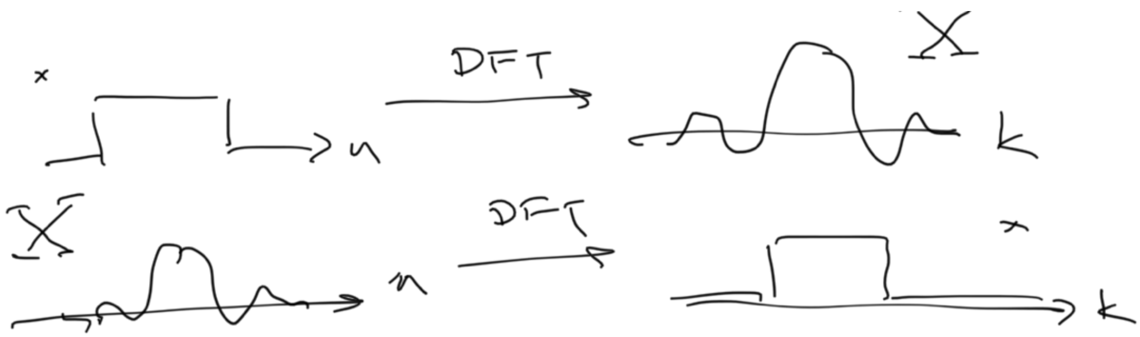
Then we obtain another analysis eqn:

$$\vec{x} = F\vec{X} \quad \text{(also analysis eqn)} \qquad (2)$$

Compare:

(1) $x[n] \xrightarrow{\text{DFT}} X[k]$

(2) $X[n] \xrightarrow{\text{DFT}} x[k]$



Q: if $F\vec{x} = \vec{X}$ and $F^*\vec{x} = \vec{X}$ for $\vec{x}, \vec{X}$ real

Does this mean $F = F^*$?

A: <u>No!</u>

$A\vec{x} = B\vec{x} \implies A = B$ only if it is true for all vectors $\vec{x}$

Here, we only consider $\vec{x}, \vec{X}$ that are real!
(But part of $\mathbb{C}^N$, so not true for all vecs)

Thus, in general $F \neq F^*$

I. LTI Systems

Linear   Time-Invariant

We are concerned w/ input-output relationships of systems:

$$\text{Input } x[n] \longrightarrow \square \longrightarrow y[n] \quad \text{Output}$$

system: circuit, optical network, computer program, digital filter bank

- LTI Systems

→ particularly well-behaved class of systems (convenient properties)

a) linearity

If $x_1[n] \longrightarrow \square \longrightarrow y_1[n]$

$x_2[n] \longrightarrow \square \longrightarrow y_2[n]$

Then,

$$x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \square \longrightarrow y[n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

superposition (additivity, scaling)

b) time invariance

"fixed behavior over time"

$$x[n-n_0] \longrightarrow \square \longrightarrow y[n-n_0]$$

If input is delayed, output is delayed the same amount

c) Checking for "LTI-ness"

Ex1: $y[n] = x[n]$

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n] = x[n]$$

Linearity:

[i] Additivity

Define $\hat{x}[n] = x_1[n] + x_2[n]$

Know $y_1[n] = x_1[n]$
$y_2[n] = x_2[n]$

$$\hat{x}[n] \longrightarrow \boxed{} \longrightarrow \hat{y}[n]$$

Check: Does $\hat{y}[n] = y_1[n] + y_2[n]$?

$$\hat{y}[n] = \hat{x}[n] = x_1[n] + x_2[n]$$

But $y_1[n] = x_1[n]$
$y_2[n] = x_2[n]$

So $x_1[n] + x_2[n] = y_1[n] + y_2[n]$

Thus, $\hat{y}[n] = y_1[n] + y_2[n]$ Additive

[ii] Scaling

Define $\hat{x}[n] = \alpha x[n]$

where $x[n] \longrightarrow \boxed{} \longrightarrow y[n]$

$$\hat{x}[n] \longrightarrow \boxed{} \longrightarrow \hat{y}[n]$$

Check: Does $\hat{y}[n] = \alpha y[n]$?

$$\hat{y}[n] = \hat{x}[n] = \alpha x[n]$$

But $y[n] = x[n]$, so $\alpha x[n] = \alpha y[n]$.

Thus, $\hat{y}[n] = \alpha y[n]$ Scaling

$\rightarrow$ Thus, **linear**

$\boxed{\text{iii}}$ Time - Invariance

Define $\hat{x}[n] = x[n-n_0]$

$$\hat{x}[n] \longrightarrow \boxed{} \longrightarrow \hat{y}[n]$$

Check: Does $\hat{y}[n] = y[n-n_0]$?

$$\hat{y}[n] = \hat{x}[n] = x[n-n_0]$$

But $y[n-n_0] = x[n-n_0]$

Thus, $\hat{y}[n] = y[n-n_0]$ Time invariant

Ex2: Not TI!

Q3, part f from today's WS

$\rightarrow y[n] = x[n] + n x[n-1]$

Let $\hat{x}[n] = x[n-n_0]$

$$\hat{x}[n] \longrightarrow \boxed{} \longrightarrow \hat{y}[n] = \hat{x}[n] + n\hat{x}[n-1]$$

Does $\hat{y}[n] = y[n-n_0]$?

$$\hat{y}[n] = x[n-n_0] + n\,x[n-n_0-1]$$

Directly substitute $n-n_0$ for n

$\neq$ NOT EQUAL

$$y[n-n_0] = x[n-n_0] + (n-n_0)\,x[n-n_0-1]$$

Thus, NOT LTI (linear, but not TI!)

Procedure

(1) Check for linearity

a) Scaling: Let $\hat{x}[n] = \alpha x[n]$

Does $\hat{y}[n] = \alpha y[n]$?

b) Additivity: Let $\hat{x}[n] = x_1[n] + x_2[n]$

Does $\hat{y}[n] = y_1[n] + y_2[n]$?

(2) Check for TI

Let $\hat{x}[n] = x[n-n_0]$

Does $\hat{y}[n] = y[n-n_0]$?

III) Convolution and Impulse Responses

$$(f*g)[n] = \sum^{\infty} f[k]\,g[n-k]$$

$k=-\infty$

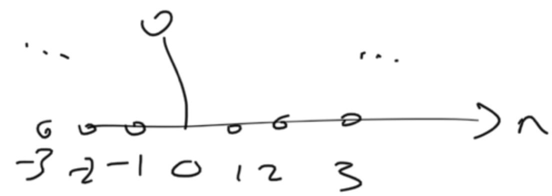
Convolution of f with g

Why do we care?

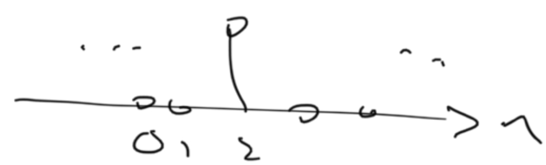
↳ Let us look at the "impulse response"

a) Impulse:

$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & \text{else} \end{cases}$$



$$\delta[n-m] = \begin{cases} 1 & n=m \\ 0 & \text{else} \end{cases}$$



$n-m=0$ when $n=m$!

b) Impulse response

$$x[n]=\delta[n] \longrightarrow \boxed{} \longrightarrow y[n]=h[n]$$

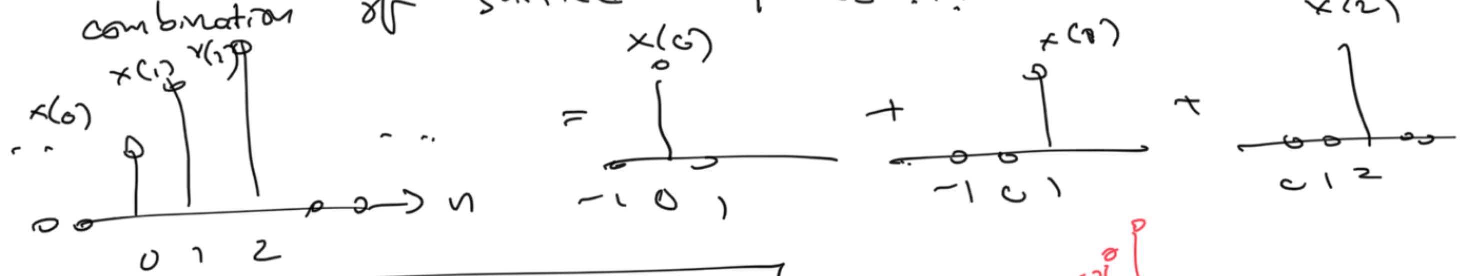
Impulse input

Impulse response

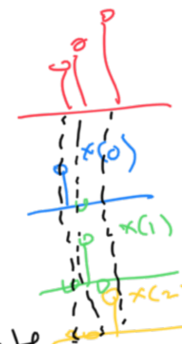
Impulse response fully characterizes the output of a system

↳ Why?

* Any input can be written as a linear combination of shifted impulses!!!



$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}$$



If input $x[n]$ into a LTI system,

$$x[n] = \sum x[k]\delta[n-k] \longrightarrow \boxed{} \longrightarrow y[n] = \sum x[k]\,h[n-k]$$

Look at $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

This is a convolution!

Thus, LTI

$$x[n] \longrightarrow \boxed{h} \longrightarrow y[n] = (x*h)[n]$$

3) Is it LTI?

(Just do a, b, c)

a) $y[n] = 4x[n]$

Linearity:

- $\hat{x}[n] = \alpha x[n]$

where $x[n] \longrightarrow \boxed{} \longrightarrow y[n] = 4x[n]$

$\hat{x}[n] \longrightarrow \boxed{} \longrightarrow \hat{y}[n] = 4\hat{x}[n]$

basically, replace all x's with $\hat{x}$

Compare to:

$\alpha y[n] = \alpha x[n]$

Does $\hat{y}[n] = \alpha y[n]$?

$\hat{y}[n] = 4\hat{x}[n] = 4\alpha x[n] = \alpha \overbrace{4x[n]}^{y[n]} = \alpha y[n]$

Thus, <u>scaling</u> satisfied

- $\hat{x}[n] = x_1[n] + x_2[n]$ substitute

$\hat{y}[n] = 4\hat{x}[n] = 4(x_1[n] + x_2[n])$

$= 4x_1[n] + 4x_2[n]$

Does $\hat{y}[n] = y_1[n] + y_2[n]$

$y_1[n] = 4x_1[n]$

$y_2[n] = 4x_2[n]$

$= y_1[n] + y_2[n]$

Thus, <u>additive</u>

Time - Invariant

$$\hat{x}[n] = x[n-n_0]$$ subs

$$\hat{y}[n] = 4\hat{x}[n] = 4x[n-n_0]$$

Does $\hat{y}[n] = y[n-n_0]$?

replace all n with $n-n_0$

$$y[n-n_0] = 4x[n-n_0]$$ equal!

Thus, $\hat{y}[n] = y[n-n_0] \Longrightarrow$ time invariant

$\boxed{LTI}$

b) $y[n] = 2x[n] - 4$

Linearity: $\hat{x}[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$$\hat{x}[n] \rightarrow \boxed{} \rightarrow \hat{y}[n] = 2\hat{x}[n] - 4$$

$$\hat{y}[n] = 2\left[\alpha_1 x_1[n] + \alpha_2 x_2[n]\right] - 4$$

$$= 2\alpha_1 x_1[n] + 2\alpha_2 x_2[n] - 4$$

Does $\hat{y}[n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$?

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] = \alpha_1(2x_1[n] - 4) + \alpha_2(2x_2[n] - 4)$$

$$= \underbrace{2\alpha_1 x_1[n]}_{ok} - 4\alpha_1 + \underbrace{2\alpha_2 x_2[n]}_{ok} - 4\alpha_2$$

not ok

$$\neq 2\alpha_1 x_1[n] + 2\alpha_2 x_2[n] - 4$$

$+ \quad 2\alpha_1 x_1 [n_0] + 2\alpha_2 x_2 [\ldots]$

$\rightarrow$ $\boxed{\text{NOT LINEAR}}$ $\longrightarrow$ Not LTI

c) $y[n] = 2x[-2+3n] + 2x[2+3n]$

- Linearity: $\hat{x}[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$$\hat{y}[n] = 2\hat{x}[-2+3n] + 2\hat{x}[2+3n]$$
$$= 2\left(\alpha_1 x_1[-2+3n] + \alpha_2 x_2[-2+3n]\right)$$
$$+ 2\left(\alpha_1 x_1[2+3n] + \alpha_2 x_2[2+3n]\right)$$

$$= 2\left(\alpha_1 x_1[-2+3n] + \alpha_1 x_1[2+3n]\right)$$
$$+ 2\left(\alpha_2 x_2[-2+3n] + \alpha_2 x_2[2+3n]\right)$$

$$\parallel$$

$$\alpha_1 y_1[n] + \alpha_2 y_2[n] = \alpha_1\left(2x_1[-2+3n] + 2x_1[2+3n]\right)$$
$$+ \alpha_2\left(2x_2[-2+3n] + 2x_2[2+3n]\right)$$

Equal! Linear!

- Time Invariance

$\hat{x}[n'] = x[n'-n_0]$

$n' = -2+3n$

$\hat{x}[n] = x[n-n_0]$

$\hat{x}[n] \longrightarrow \boxed{} \longrightarrow \hat{y}[n] = 2\hat{x}[-2+3n] + 2\hat{x}[2+3n]$

$$= 2x[-2+3n-n_0] + 2x[2+3n-n_0]$$

Does $\hat{y}[n] = y[n-n_0]$?

To calculate $y[n-n_0]$, replace all n's with $n-n_0$

$$y[n-n_0] = 2x[-2+3(n-n_0)] + 2x[2+3(n-n_0)]$$

$$= 2x[-2+3n-3n_0] + 2x[2+3n-3n_0]$$

not equal!

Not $\underbrace{\text{time-invariant}}$!

$\Rightarrow$ Not LTI

d) Not LTI (nonlinear)

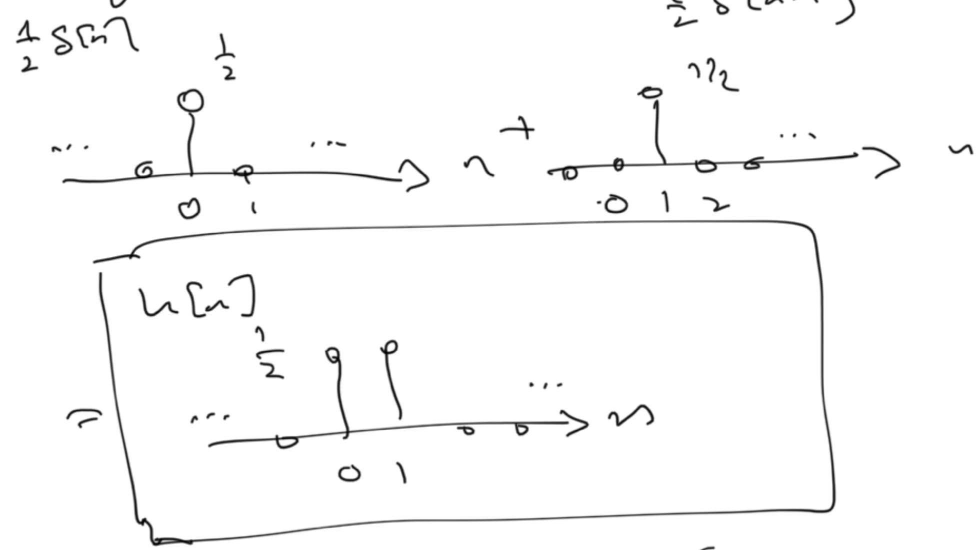
e) LTI $\longrightarrow$ google LCCDEs (EE120 material more likely)

f) Not LTI (time-varying)

5) Mystery System

Impulse response: $h[n] = \frac{1}{2}[\delta[n] + \delta[n-1]]$

a) Sketch the impulse response $h[n]$



$$h[0] = \frac{1}{2}[\delta[0] + \cancel{\delta[0-1]}] = \frac{1}{2}\, 1 = \frac{1}{2}$$

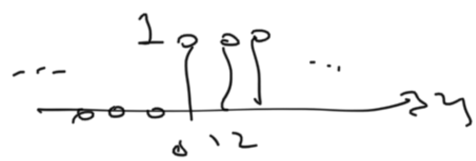
$$h[1] = \frac{1}{2}\left[\underbrace{\cancel{\delta[1]}}_{0} + \underbrace{\delta[1-1]}_{1}\right] = \frac{1}{2}\,1 = \frac{1}{2}$$

$$h[n>1] = \frac{1}{2}\left[\underbrace{\delta[n>1]}_{=0} + \underbrace{\delta[n-1>0]}_{=0}\right] = 0$$

$$h[n<0] = \frac{1}{2}\left[\underbrace{\cancel{\delta[n<0]}}_{=0} + \underbrace{\cancel{\delta[n-1<-1]}}_{=0}\right] = 0$$

b) input is a unit step $u[n]$:

$$u[n] \equiv \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$u[n] \longrightarrow \boxed{h} \longrightarrow y[n] = u[n] * h[n]$$

$$\rightarrow y[n] = \sum_{k=-\infty}^{\infty} u[k]\,h[n-k] = \sum_{k=0}^{\infty} h[n-k]$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\delta[n-k] + \delta[n-k-1]\right)$$

$n<0$: $n-k<0 \Longrightarrow h[n-k] = 0$

$$y[n<0] = \frac{1}{2}\sum_{k=0}^{\infty} \underbrace{\cancel{\delta[n-k \geq 0]}}_{0} + \underbrace{\cancel{\delta[n-k-1<-1]}}_{0}$$

$= 0$

$n=0$: $$y[0] = \frac{1}{2}\sum_{k=0}^{\infty} \delta[-k] + \delta[\underbrace{-k-1}_{<-1}]$$

$\delta[-k] = 1$ if $k=0$

$= \frac{1}{2}$

$$= 2$$

$n = 1$: $$y[1] = \frac{1}{2} \sum_{k=0}^{\infty} \left(\delta[k-k] + \delta[1-k-1] \right)$$

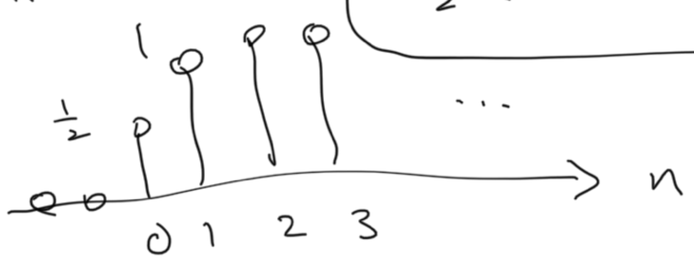
$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\underbrace{\delta[0]}_{1} + \underbrace{\delta[-k]}_{=1 \text{ if } k=0} \right)$$

$$= \frac{1}{2}(1+1) = 1$$

$n \geq 1$: $$y[n] = \frac{1}{2} \sum_{k=0}^{\infty} \left(\underbrace{\delta[n-k]}_{=1 \text{ for } k=n\geq 1} + \underbrace{\delta[n-k-1]}_{=1 \text{ for } k=n-1\geq 0} \right)$$

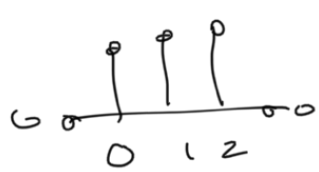
$$= \frac{1}{2}(1+1) = 1$$

$y[n] = u[n] * h[n]$

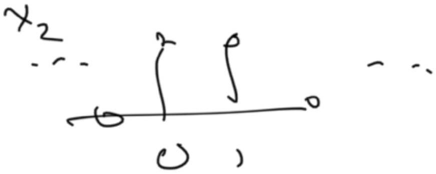


Graphically, convolution is "flip and slide"

x_1



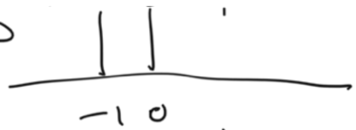
$*$ x_2



Flip x_2,

take dot product

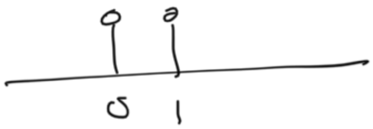
$n=0$



$\longrightarrow \; 0\times1 + 1\times1 + 1\times0 - 1\times0$

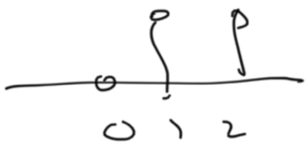
$(x_1 * x_2)(0) = 1$

$n=1$



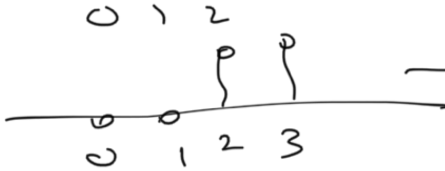
$\longrightarrow \; 1\times1 + 1\times1 + 1\times0 = 2$

$n=2$



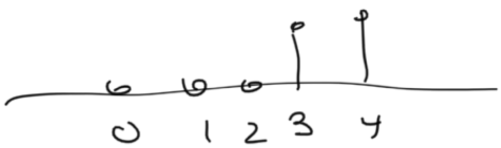
$\longrightarrow \; 1\times0 + 1\times1 + 1\times1 = 2$

$n=3$



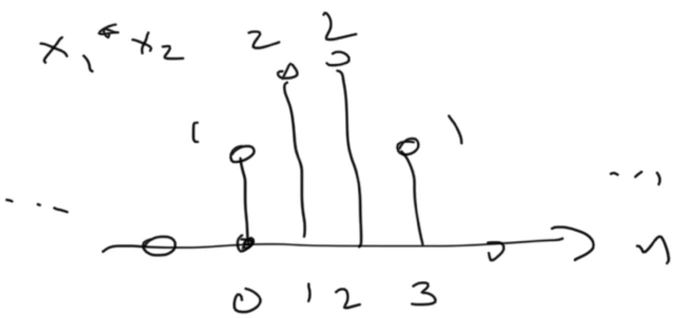
$\longrightarrow \; 0\times1 + 0\times1 + 1\times1 = 1$

$n=4$:

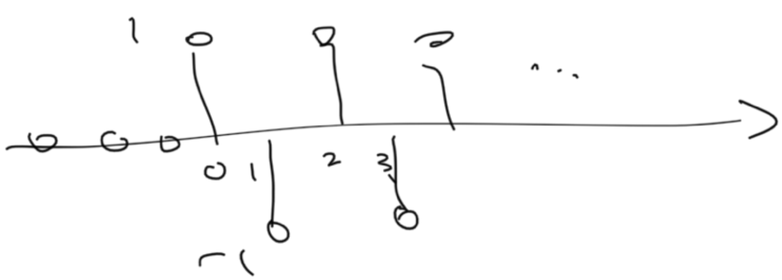


$\longrightarrow \; 0$

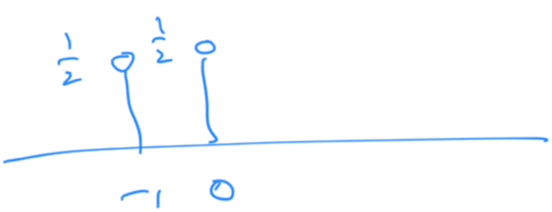
$x_1 * x_2$



c) $x[n] = (-1)^n u[n]$

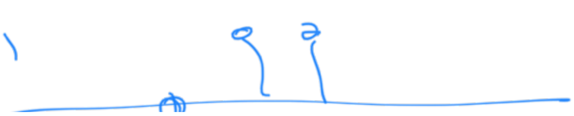


$n=0$



$\longrightarrow \; \frac{1}{2}$

$n=1$



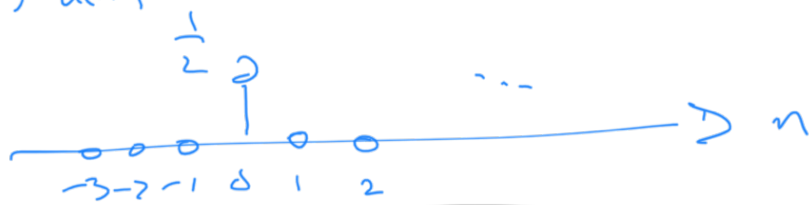
$\longrightarrow \; \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$

$n = 2$

$$\frac{1}{2}(-1) + \frac{1}{2}(1) = 0$$

$$\frac{-1+1}{2}$$

$$\boxed{(-1)^n u[n] * h[n]}$$



d) SMA?

$$y[n] = \frac{x[n] + x[n-1]}{2}$$

average of current and the last timestep!

→ takes a moving average of two time steps

→ e.g. stock market analysis

(4) Convoluted Convolution

a) Show convolution is commutative

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Define $m = n-k$, $k = n-m$

$$= \sum_{m=-\infty}^{\infty} x[n-m]\, h[m]$$

$$= \sum_{m=-\infty}^{\infty} h[m]\, x[n-m]$$

$$= (h * x)[n] \qquad \square$$

b) Show $(x * \delta)[n] = x[n]$

($\delta[n]$ is convolution identity)

$$(x * \delta)[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k] = x[n] \qquad \square$$

selects $n=k$

c) Time-shift. Show $x * \delta[n-n_0] = x[n-n_0]$

$$x[n] * \delta[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-n_0-k]$$

selects $k = n-n_0$

$$= x[n-n_0] \qquad \square$$

d) Distributivity: Show $x * (h_1+h_2) = x * h_1 + x * h_2$

$$(x * h_1+h_2)[n] = \sum x[k] \left(h_1[n-k] + h_2[n-k] \right)$$

$$= \sum \left(x[k]\, h_1[n-k] + x[k]\, h_2[n-k] \right)$$

$$= \sum x[k]\, h_1[n-k] + \sum x[k]\, h_2[n-k]$$

$$= x * h_1 + x * h_2 \qquad \square$$

Bonus: DFT Summary

Synthesis

$$\vec{x} = F^* \vec{X} = \sum_{k=0}^{N-1} X[k]\, \vec{u}_k$$

$$\ldots = \frac{1}{N} \sum^{N-1} x[n]\, e^{j\frac{2\pi}{N}kn}$$

Equation $\quad x[n] = \frac{1}{\sqrt{N}} \sum_{k=0} X[k]\, \omega_N \qquad \frac{1}{\sqrt{N}} \sum_{k=0}$

Analysis Equation

$$\vec{X} = F\vec{x} = \sum_{k=0}^{N-1} x[k]\, \overline{\vec{u}_n}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\, \omega_N^{-kn} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$F^* = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & & | \end{bmatrix} \qquad u_k[n] = \omega_N^{kn}$$

$$F = \begin{bmatrix} | & & | \\ \overline{\vec{u}_1} & \cdots & \overline{\vec{u}_{N-1}} \\ | & & | \end{bmatrix} \qquad \overline{u_k[n]} = \omega_N^{-kn}$$

* Common Transform Pairs

Time Domain		Freq Domain
$x[n] = \cos\left(\frac{2\pi}{N}mn\right)$	$\longleftrightarrow$	$X[k] = \begin{cases} \frac{\sqrt{N}}{2} & k=m, N-m \\ 0 & \text{else} \end{cases}$
$x[n] = \sin\left(\frac{2\pi}{N}mn\right)$	$\longleftrightarrow$	$X[k] = \begin{cases} \frac{\sqrt{N}}{2j} & k=m \\ -\frac{\sqrt{N}}{2j} & k=N-m \\ 0 & \text{else} \end{cases}$
$x[n] = 1$ (constant)	$\longleftrightarrow$	$X[k] = \sqrt{N}\,\delta[k] = \begin{cases} \sqrt{N} & k=0 \\ 0 & \text{else} \end{cases}$
$x[n] = e^{j\frac{2\pi}{N}mn}$	$\longleftrightarrow$	$X[k] = \sqrt{N}\,\delta[k-m] = \begin{cases} \sqrt{N} & k=m \\ 0 & \text{else} \end{cases}$
$x[n] = \delta[n]$	$\longleftrightarrow$	$X[k] = \frac{1}{\sqrt{N}}$

$$x[n] = \delta[n-m] \quad \longleftrightarrow \quad X[k] = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}mk}$$

$$x[n] = \text{boxcar} \quad \longleftrightarrow \quad X[k] = \text{sinc}$$

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