Discussion 8A

Circular Convolution and Aliasing

* Circulant Matrices
  - Derivation: DFT Basis Diagonalize Circulant Matrices
* Circular Convolution and DFT
  - Derivation: Circular Convolution in Time is Multiplication in Frequency
* Sampling and Aliasing

I. Circulant Matrices

Consider a matrix as follows:

$$C_h = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & \cdots & h_2 \\ & & \vdots & & \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix} \quad \leftarrow N \times N \text{ matrix}$$

Each row is rotated one element to the right (as we go down the rows)

$$\text{If} \quad \vec{h} = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{N-1} \end{bmatrix} \quad \leftarrow \text{column vector, defines } C_h$$

$$\text{DFT} \quad \vec{h} = \begin{bmatrix} H[0] \end{bmatrix}$$

has DFT $\vec{H} = \begin{bmatrix} H[0] \\ \vdots \\ H[N-1] \end{bmatrix}$

Then,

$$\boxed{C_h \vec{u}_k = \left(\sqrt{N}\, H[k]\right) \vec{u}_k}$$

eigenvec eigenval

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega_N^{k \cdot 0} \\ \omega_N^{k \cdot 1} \\ \vdots \\ \omega_N^{k \cdot N-1} \end{bmatrix}$$

vector

$$u_k[n] = \frac{1}{\sqrt{N}} \omega_N^{kn}$$

In other words, C_h is diagonalized by the DFT basis!

$$C_h = U \Lambda_H U^* \quad \left(U = \left[\vec{u}_0 \; \cdots \; \vec{u}_{N-1} \right] \right)$$

$$= F^* \Lambda_H F \quad \left(F = U^* \right)$$

$$\boxed{C_h = F^* \begin{bmatrix} \sqrt{N}\, H[0] & & & 0 \\ & \sqrt{N}\, H[1] & & \\ & & \ddots & \\ 0 & & & \sqrt{N}\, H[N-1] \end{bmatrix} F}$$

Proof?

$$h[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} H[k]\, \omega_N^{kn}$$ ← DFT synthesis equation (summation)

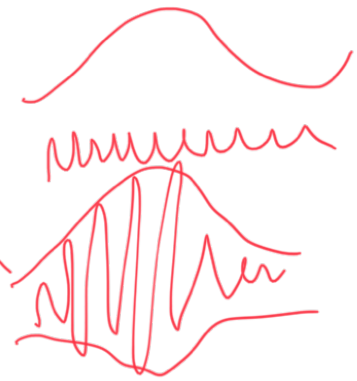
$$h[n-n_0] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} H[k]\, \omega_N^{k(n-n_0)}$$

circular shift

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \underbrace{H[k]\, \omega_N^{-kn_0}}_{\hat{H}[k]}\, \omega_N^{kn}$$

(0 shift)

modulation



$$= \frac{1}{\sqrt{N}} \sum \hat{H}[k]\, \omega_N^{kn}$$

↳ DFT coefficients of $h[n-n_0] = \hat{h}[n]$

are $\hat{H}[k] = H[k]\, \omega_N^{-kn_0}$

⟹ circular shifts in time are modulation in frequency!

$$\underset{(\text{mod } N)}{h[n-n_0]} \overset{DFT}{\Longleftrightarrow} \underline{H[k]\, \omega_N^{-kn_0}}$$

Thus,

$$C_h = \begin{bmatrix} | & | & & | \\ \vec{h}_0 & \vec{h}_1 & \cdots & \vec{h}_{N-1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} h_0 & h_{N-1} & & h_1 \\ h_1 & h_0 & \cdots & h_2 \\ & h_1 & & \\ \vdots & \vdots & & \vdots \\ h_{N-1} & h_{N-2} & & h_0 \end{bmatrix}$$

h_1 indicates a circular shift of h

n_0 indicates a shift of n_0,
so h_0 has 0 shift

Note that if $h[n-n_0 \pmod N] \xleftrightarrow{DFT} \underline{1\, H[k]\, \omega_N^{-kn_0}}$,

then $F\vec{h}_{n_0} = \sqrt{N}\,\vec{H} \odot \vec{u}_{n_0} = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega_N^{-0 \cdot n_0} \\ \vdots \\ \omega_N^{-(N-1) \cdot n_0} \end{bmatrix}$ element wise multiplication

$$= \sqrt{N}\frac{1}{\sqrt{N}} \begin{bmatrix} H[0]\,\omega_N^{-0 \cdot n_0} \\ H[1]\,\omega_N^{-1 \cdot n_0} \\ \vdots \\ H[N-1]\,\omega_N^{-(N-1 \cdot n_0)} \end{bmatrix}$$

$$C_h = \begin{bmatrix} \vec{h}_0 & \vec{h}_1 & \cdots & \vec{h}_{N-1} \end{bmatrix} = \begin{bmatrix} \sqrt{N}F^*(\vec{H}\odot\vec{u}_0) & \sqrt{N}F^*(\vec{H}\odot\vec{u}_1) & \cdots & \sqrt{N}F^*(\vec{H}\odot\vec{u}_{N-1}) \end{bmatrix}$$

$$= F^* \begin{bmatrix} \sqrt{N}\vec{H}\odot\vec{u}_0 & \sqrt{N}\vec{H}\odot\vec{u}_1 & \cdots & \sqrt{N}\vec{H}\odot\vec{u}_{N-1} \end{bmatrix}$$

$$= F^* \begin{bmatrix} \sqrt{N}H[0] & & & \\ & \sqrt{N}H[1] & & \\ & & \ddots & \\ & & & \sqrt{N}H[N-1] \end{bmatrix} \underbrace{\begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}}_{F}$$

$$= F^* \begin{bmatrix} \sqrt{N}H[0] & & 0 \\ & \sqrt{N}H[1] & \\ 0 & \ddots & \sqrt{N}H[N-1] \end{bmatrix} F$$

$$= F^* \Lambda_H F$$

II. Circular Convolution and DFT

From lecture, we can describe the input output relationship of a periodic discrete-time LTI system with

$$\boxed{\begin{array}{c} \vec{y} = C_h \vec{x} \\ \text{circular convolution} \end{array}}$$

Remember, convolution is important because it allows us to calculate the output of a LTI system!

$$x \longrightarrow \boxed{h} \longrightarrow y = x * h$$

impulse response

only true for LTI systems!

We will show:

$$\boxed{\begin{array}{c} \text{“Circular Convolution in time is element-wise multiplication in frequency”} \\ x[n] \circledast h[n] \xleftrightarrow{DFT} \sqrt{N}\, \vec{X} \odot \vec{H} \end{array}}$$

Proof:

$\vec{y} = C_h \vec{x} = x[n] \circledast h[n]$ circular

$y \quad h$

convolution

$$S \stackrel{?}{=} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & 0 & \cdots & & 0 \end{pmatrix}$$

$$S \begin{pmatrix} h_0 \\ \vdots \\ h_{N-1} \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{N-1} \\ h_0 \end{pmatrix}$$

where $C_h \vec{\delta}_0 = \vec{h}$

$$\delta_k[n] = \begin{cases} 1 & n = k \\ 0 & \text{else} \end{cases}$$

$$\vec{\delta}_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

circularly shifted!!

Then $$C_h = \begin{bmatrix} | & | & & | \\ \vec{h}_0 & \vec{h}_1 & \cdots & \vec{h}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\vec{y} = C_h \vec{x} = F^* \Lambda_H F \vec{x}$$

$$\Rightarrow \vec{Fy} = \Lambda_H F \vec{x}$$

But, $\vec{Y} = F\vec{y}$, $\vec{X} = F\vec{x}$

So $$\vec{Y} = \Lambda_H \vec{X}$$

$$= \sqrt{N} \begin{bmatrix} H[0] & & & \\ & H[1] & & \\ & & \ddots & \\ & & & H[N-1] \end{bmatrix} \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$= \sqrt{N} \begin{bmatrix} H[0]\,X[0] \\ H[1]\,X[1] \\ \vdots \\ H[N-1]\,X[N-1] \end{bmatrix}$$

Or, $\vec{Y} = \sqrt{N}\,\vec{H} \odot \vec{X}$ ← vectors

kth element (0 indexed) → $Y[k] = \sqrt{N}\,H[k]\,X[k]$ ← scalars

In Q1 of Dis 8A,

what if our system is not periodic?

⟹ zero-pad both $\vec{x}$ and $\vec{h}$, where $\vec{x}$ is length N and $\vec{h}$ is length M, to length $M+N-1$.

Procedure

(i) Compute DFT of $\vec{x}$ and $\vec{h}$ (where they are appropriately zero-padded if not a DT periodic LTI system)

$$\underbrace{\vec{X} = F\vec{x}, \quad \vec{H} = F\vec{h}}$$

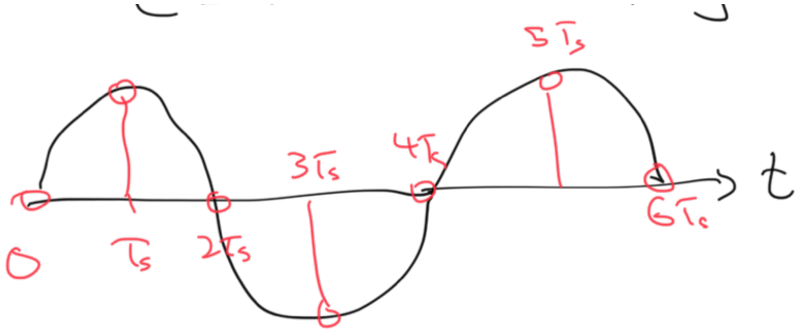
(ii) $$\underbrace{\vec{Y} = \sqrt{N}\,\vec{X} \odot \vec{H}}$$

(iii) Inverse DFT to get $\vec{y}$

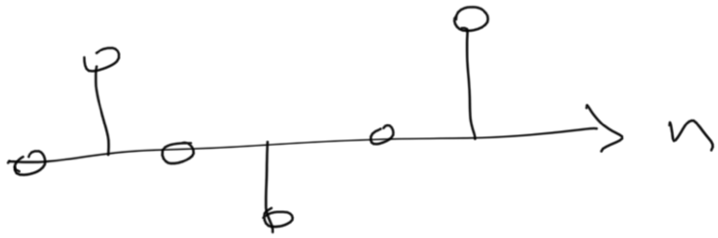
$$\underbrace{\vec{y} = F^{*}\,\vec{Y}}$$

III. Sampling Theorem

(Intro to sampling + aliasing)



Sample w/ some period T_s

$\Rightarrow$ 

How to reconstruct?

reconstruct via linear combination of shifted basis functions

$$\boxed{\begin{array}{l}\text{Main way: "sinc interpolation"} \\ \text{Reconstructed signal:} \quad \hat{x}(t) = \sum x[n]\, \Phi(t - nT_s) \\ \qquad\qquad\qquad\qquad\quad \Phi(t) = \text{sinc}\left(\frac{t}{T_s}\right)\end{array}}$$

Will $\hat{x}$ always perfectly reconstruct the original signal $x(t)$, i.e. will $\hat{x}(t) = x(t)$?

$\Rightarrow$ NO!

Depends on sampling frequency / period

— need to sample "fast enough" to capture all variations in the signal

...

perfectly

If $T_s \equiv$ sampling period (s)

$\omega_s = \frac{2\pi}{T_s} \equiv$ sampling frequency $\left(\frac{rad}{s}\right)$

Nyquist-Shannon Sampling Theorem

If the highest frequency present in the original signal is ω_{max}, then to perfectly reconstruct we require:

$$\boxed{\omega_s > 2\omega_{max}}$$

Mnemonic: human hearing $\sim$ 20 - 20 kHz

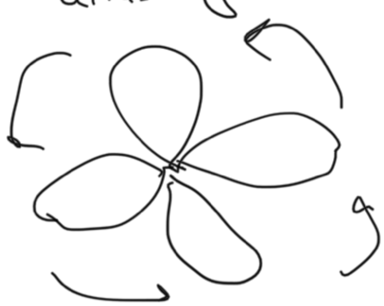
$\Rightarrow f_{max} = 20$ kHz

Music CDs (?) sample at $f_s = 44.1$ kHz

$\Rightarrow f_s \approx 2 f_{max}$

Recover some other signal if $\omega_s \leq 2\omega_{max}$

$\Longrightarrow$ aliasing



(helicopter video — see today's lecture)

"[illegible] wheel effect"

carriage wheel effect

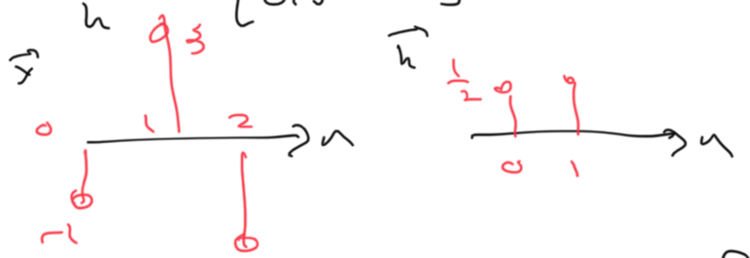
Q1, Q4

DFT and convolution — aliasing

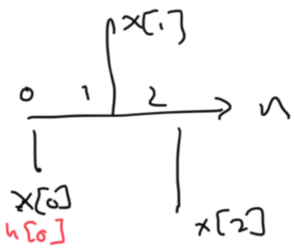
1 Circulant Matrices and Convolution

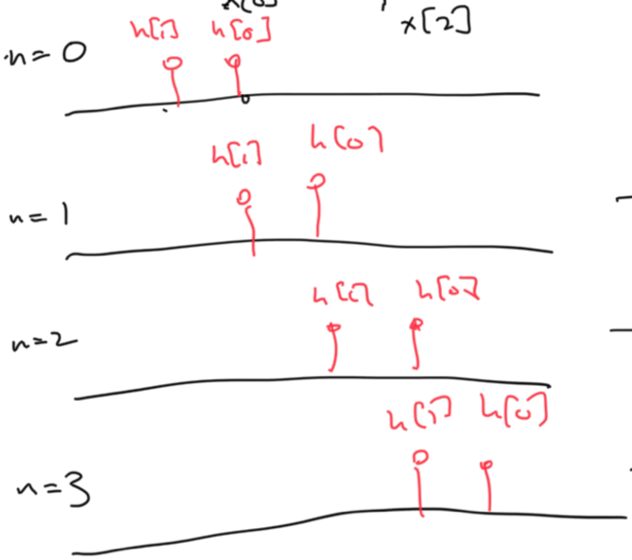
$$\vec{x} = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}^T$$

$$\vec{h} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T$$



a) Calculate $x[n] * h[n]$. What is the length of the output?





$$y[0] = h[0]x[0] = -0.5$$

$$y[1] = h[1]x[0] + h[0]x[1] = 1$$

$$y[2] = h[1]x[1] + h[0]x[2] = 0.5$$

$$y[3] = h[1]x[2] = -1$$

$$\vec{y} = \begin{bmatrix} -0.5 & 1 & 0.5 & -1 \end{bmatrix}^T$$

$\vec{x}$: $M = 3$ m length

$\vec{h}$: $N=2$ in length

$\longrightarrow$ $\vec{y}$ is $M+N-1 = 4$ in length

If $\vec{x}$ is length M and $\vec{h}$ is length N,

$y = x * h$ is length $M+N-1$

$\Rightarrow$ zeropad both x and h to be this length

b) As a matrix multiplication?

$$\underbrace{\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ 0 & h[1] & h[0] \\ 0 & 0 & h[1] \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix}}_{\vec{x}}$$

Almost circulant ... need to zeropad

c) Zeropad!

$\vec{h} = \left[\frac{1}{2} \;\; \frac{1}{2} \;\; 0 \;\; 0 \right]$

$\vec{x} = \left[-1 \;\; 3 \;\; -2 \;\; 0 \right]$

$$\vec{y} = \begin{bmatrix} h[0] & 0 & 0 & h[1] \\ h[1] & h[0] & 0 & 0 \\ 0 & h[1] & h[0] & 0 \\ 0 & 0 & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ 0 \end{bmatrix}$$

$$C_h = \begin{bmatrix} \vec{h}_0 & \vec{h}_1 & h_2 & h_3 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}$$

circular shifts of $\vec{h}$

d) Calculate $\vec{y}$ using the DFT instead of convolution

$\boxed{i}$ $\vec{X} = F\vec{x}$, $\vec{H} = F\vec{h}$

$$F^* = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}, \quad F = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

How? $\vec{u}_0, \vec{u}_1, \vec{u}_2, \vec{u}_3$ for $N=4$ (length 4 signals)

$$\vec{u}_0 = \frac{1}{\sqrt{4}} \begin{bmatrix} \omega_4^{0\cdot 0} \\ \omega_4^{0\cdot 1} \\ \omega_4^{0\cdot 2} \\ \omega_4^{0\cdot 3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\omega_4 = e^{j\frac{2\pi}{4}} = e^{j\frac{\pi}{2}} = j$$

$$\vec{u}_1 = \frac{1}{\sqrt{4}} \begin{bmatrix} \omega_4^{1\cdot 0} \\ \omega_4^{1\cdot 1} \\ \omega_4^{1\cdot 2} \\ \omega_4^{1\cdot 3} \end{bmatrix} = \frac{1}{\sqrt{4}} \begin{bmatrix} j^0 \\ j^1 \\ j^2 \\ j^3 \end{bmatrix} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{4}} \begin{bmatrix} (-1)^0 \\ (-1)^1 \\ (-1)^2 \\ (-1)^3 \end{bmatrix} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{4}} \begin{bmatrix} (-j)^0 \\ (-j)^1 \\ (-j)^2 \\ (-j)^3 \end{bmatrix} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix}$$

$\vec{X} = \begin{bmatrix} 0 \\ \ldots \end{bmatrix}$ $\quad \vec{H} = \begin{bmatrix} 0.5 \\ \ldots \end{bmatrix}$

$$\vec{X} = \begin{bmatrix} 0.5 - 1.5j \\ -3 \\ 0.5 + 1.5j \end{bmatrix} \qquad \vec{H} = \begin{bmatrix} 0.25 - 0.25j \\ 0 \\ 0.25 + 0.25j \end{bmatrix}$$

(ii) $\vec{Y} = \sqrt{4}\,\vec{H} \odot \vec{X}$

$$= 2 \begin{bmatrix} 0.5 \\ 0.25 - 0.25j \\ 0 \\ 0.25 + 0.25j \end{bmatrix} \odot \begin{bmatrix} 0 \\ 0.5 - 1.5j \\ -3 \\ 0.5 + 1.5j \end{bmatrix}$$

$$= 2 \begin{bmatrix} 0.5 \cdot 0 \\ (0.25 - 0.25j)(0.5 - 1.5j) \\ 0 \cdot -3 \\ (0.25 + 0.25j)(0.5 + 1.5j) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -0.5 - j \\ 0 \\ -0.5 + j \end{bmatrix}$$

(iii) $\boxed{\vec{y} = F^* \vec{Y} = \begin{bmatrix} -0.5 \\ 1 \\ 0.5 \\ -1 \end{bmatrix}}$ Same as before!

e) Why is this important?

FFT: $O(N \log N)$

Convolution: $O(N^2)$

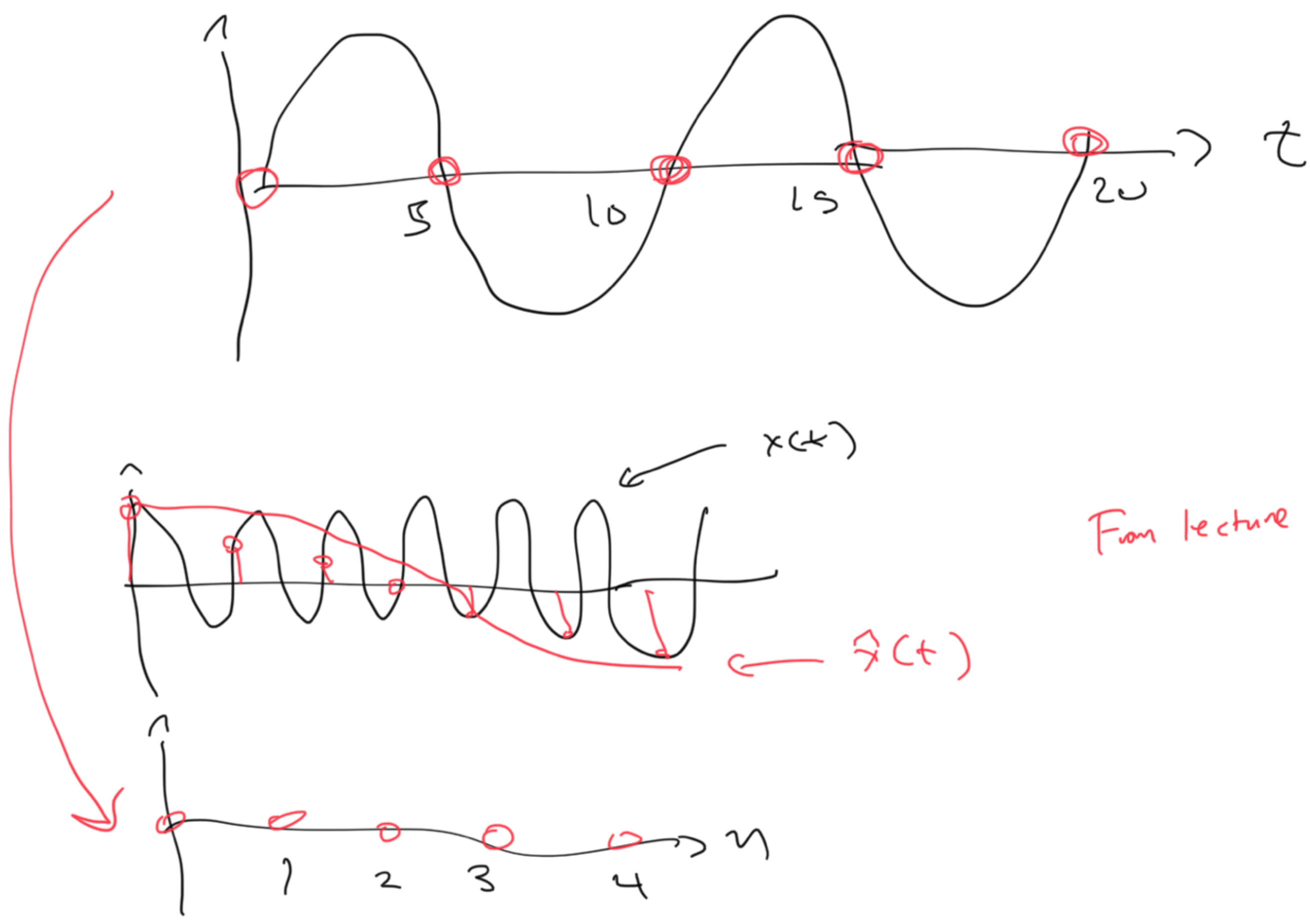
Ex: 10^6 elements $\rightarrow$ 10^{12} operations (direct convolution)

$\rightarrow$ 6×10^6 operations (FFT)

④ Aliasing

$x(t) = \sin(0.2\pi t)$

a) What would the sampling period T_s be to recover a constant at zero?



$$\boxed{T = 5\ s}$$

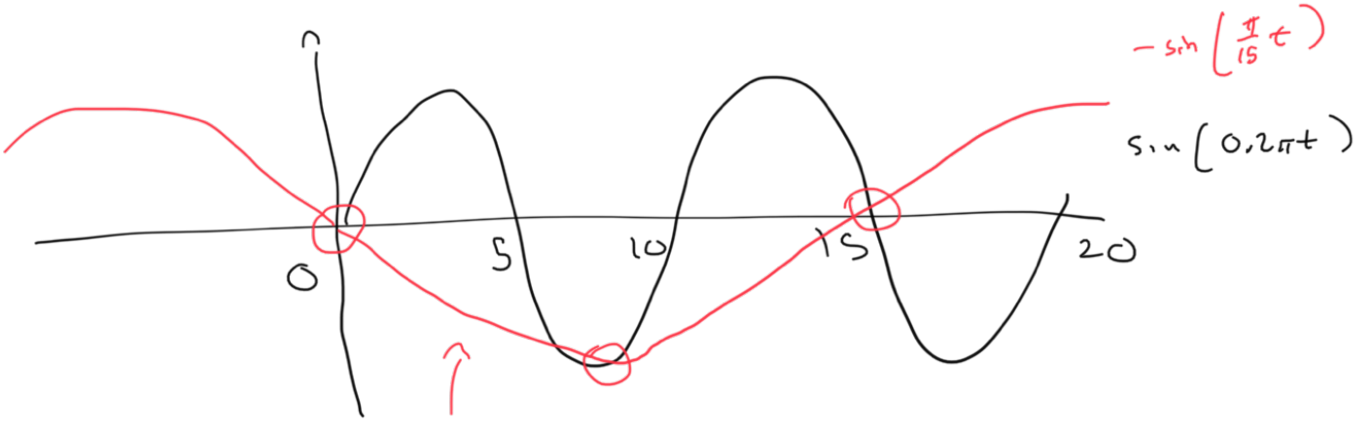
$\sin(0.2\pi t) = 0$ if $0.2\cancel{\pi} t = m\cancel{\pi}$

$t = 5m, \ m \in \mathbb{Z}$

$\Rightarrow$ sample at $T_s = 5$

$\left(\frac{\pi}{10} t\right)$?

b) Recover $\hat{x}(t) = -\sin(\frac{\pi}{15}t)$



Look at intersection points — every 7.5 seconds

$$\Rightarrow \boxed{T_s = 7.5\ s}$$

$$x[n] = x(nT) = \hat{x}(nT)$$

$$x(t) = \sin(0.2\pi t)$$

$$\hat{x}(t) = -\sin\left(\frac{\pi}{15}t\right)$$

$$\Rightarrow \sin(0.2\pi nT) = -\sin\left(\frac{\pi}{15}nT\right)$$

$$= -\sin(-0.2\pi nT)$$

$$= -\sin(-0.2\pi nT + 2\pi n)$$

$$\sin(x) = \sin(x + 2\pi)$$
$$= \sin(x + 2\pi n)$$

Thus, $-\sin(-0.2\pi nT + 2\pi n) = -\sin\left(\frac{\pi}{15}nT\right)$

True for all n, so

$$-0.2\pi \cancel{n} T + 2\pi \cancel{n} = \frac{\pi}{15} \cancel{n} T$$

$$-0.2\pi T + 2\pi = \frac{\pi}{15} T$$

$$2\pi = \frac{4\pi}{15} T$$

$$\boxed{T = 7.5}$$

Alias at $\omega \pm 2\pi i$, $i \in \mathbb{Z}$

2) Sampling Theorem Basics

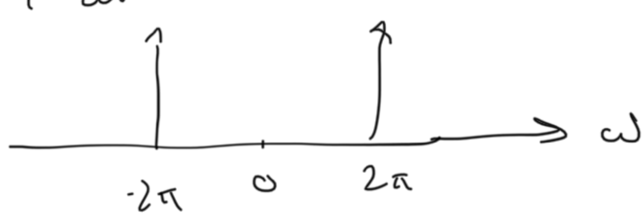
$$x(t) = \cos(2\pi t)$$

a) $\cos(\omega t) = \cos(2\pi t)$

$$\Longrightarrow \boxed{\begin{aligned} \omega &= 2\pi \text{ rad/s} = \omega_{max} \\ f &= \frac{\omega}{2\pi} = 1 \text{ Hz} = f_{max} \end{aligned}}$$

not DFT!

(FT would look like this:)



b) Sample every T seconds:

$$\boxed{\omega_s = \frac{2\pi}{T}}$$ by definition

c) Smallest T w/ imperfect reconstruction:

$\omega_s > 2\omega_{max}$ for <u>perfect</u> reconstruction

$\Longrightarrow \omega_s \leq 2\omega_{max}$ for <u>imperfect</u>

Thus, $\frac{2\pi}{T} \leq 2\omega_{max} \Longrightarrow T \geq \frac{\pi}{\omega_{max}}$

If $\omega_{max} = 2\pi$,

$T \geq \frac{\pi}{2\pi} = \frac{1}{2}$ s

$$\boxed{T_{min} = \frac{1}{2} \text{ s}}$$

(3) More Sampling

$T_m = \frac{1}{4}$ s, $T_n = 1$ s

$\Downarrow$ reconstructs f_m

$\Downarrow$ reconstructs f_n

original signal:

$x(t) = \cos(2\pi t)$

a) Nyquist limit satisfied?

$\omega_s > 2\omega_{max} = 2(2\pi)$

If $\omega_s = \frac{2\pi}{T}$, $\frac{\cancel{2\pi}}{T} > 2(\cancel{2\pi}) \Longrightarrow T < \frac{1}{2}$ s

$T_m < \frac{1}{2}$ s	✓	satisfied
$T_n > \frac{1}{2}$ s	✗	not satisfied

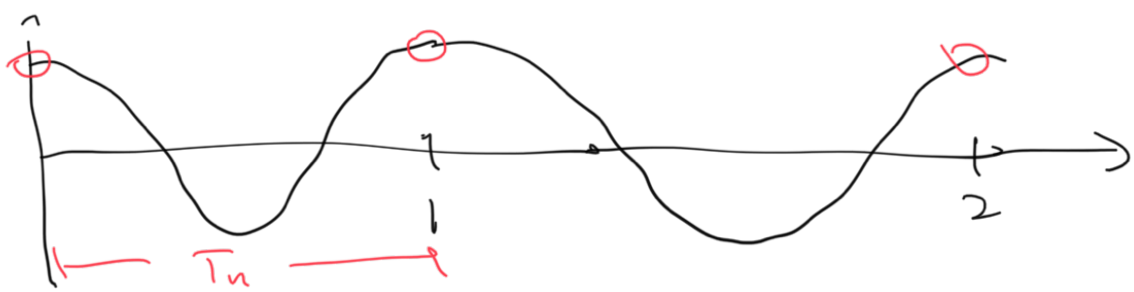
b) Reconstruct w/ T_n?

Then $\omega_{s,n} = \frac{2\pi}{T} = 2\pi$ rad/s

$2\omega_{max} < \omega_{s,n}$

$\Rightarrow \boxed{\omega_{max} = \frac{\omega_{s,n}}{2} = \frac{2\pi}{2} = \pi \text{ rad/s}}$

c) Periodic functions w/ freq $\omega \leq \pi$, but with same samples?



Just constant samples of 1!

$\boxed{\text{Reconstruct} \quad f_n(t) = 1}$