Discussion 8B

Signals Review

- Review
  - Roots of Unity
  - DFT
  - LTI Systems
  - Convolution
  - Circulant Matrices
  - Sampling Theorem
- Questions
  - Dis 7C Q1 or Signals Review WS Q2
  - Sp 16 Final Q5
  - HW8 Q3
  - HW8 Q4

(I.) Roots of Unity

- A root of unity is a solution to $z^N = 1$
- In general, there are N unique roots of unity

$$\rightarrow 1, e^{j\frac{2\pi}{N}}, e^{j\frac{2\pi}{N}2}, \ldots, e^{j\frac{2\pi}{N}(N-1)}$$

$\underbrace{\hphantom{xxxxxxxxxxxxxxxx}}_{N}$

In general, can write as:

$$\omega_N^k = e^{j\frac{2\pi}{N}k}$$

$$\omega_N = e^{j\frac{2\pi}{N}} \quad (\text{sometimes written as } \omega)$$

Note that roots of unity are N-cyclic:

i.e. $\omega_N^k = \omega_N^{k+N} = \omega_N^{k-N} = \cdots$

So if we want to consider only $k = 0, 1 \ldots N-1$

$\Longrightarrow$ consider $\underline{k \bmod N}$

EX: $\omega_4^{-1} = \omega_4^{-1 \bmod 4} = \omega_4^3$

1
=

Explicitly, $\omega_4^{-1} = e^{+j\frac{2\pi}{N}(-1)} = e^{j\frac{2\pi}{N}(-1)} e^{j\frac{2\pi}{N}N}$

$= e^{j\frac{2\pi}{N}(-1+4)} = e^{j\frac{2\pi}{N}3} = \omega_4^3$

II. DFT

The DFT is a change of basis into a basis where each basis element $\vec{u}_k$ represent a pure frequency.

The DFT coefficients $X[k]$ tell us how much of the k^{th} frequency we have in our signal

kth root of unity

- DFT Basis

$$u_k[n] = \frac{1}{\sqrt{N}} \omega_N^{kn} = \frac{1}{\sqrt{N}} (\omega_N^k)^n = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn}$$

↳ n^{th} element (0 indexed) of the k^{th} basis vector

$$\vec{u}_k = [u_k[0] \;\; \ldots \;\; u_k[N-1]]^T$$

$$F^* = \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$F = \begin{bmatrix} | & | & & | \\ \overline{\vec{u}_0} & \overline{\vec{u}_1} & \cdots & \overline{\vec{u}_{N-1}} \\ | & | & & | \end{bmatrix}$$

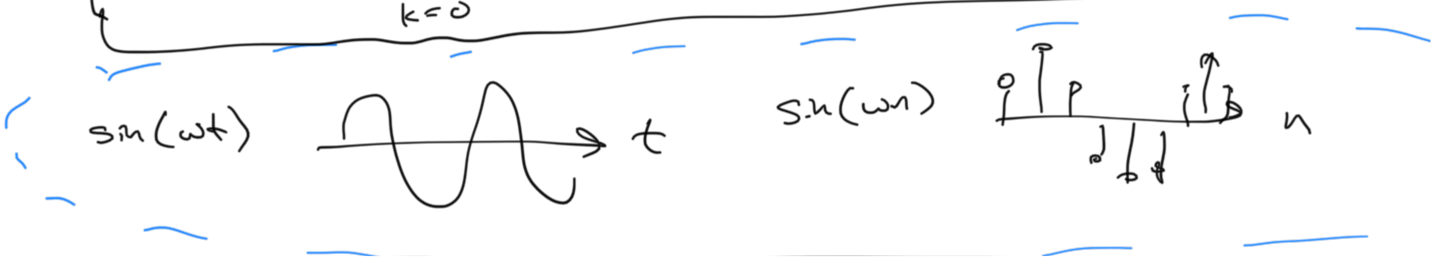
- unitary
- symmetric] ⟶ $F^{-1} = F^* = \overline{F}$
- linear

- Equations

$$\vec{x} = F^* \vec{X} = \sum_{k=0}^{N-1} X[k]\, \vec{u}_k$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

} Synthesis



$$\vec{X} = F\vec{x} = \sum_{n=0}^{N-1} x[n]\,\vec{u}_n$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

} Analysis

- Common Transform Pairs

<u>Time (n)</u> | <u>Frequency (k)</u>

$$x[n] = \cos\left(\frac{2\pi}{N}mn\right) \longleftrightarrow X[k] = \begin{cases} \frac{\sqrt{N}}{2} & k = m, N-m \\ 0 & \text{else} \end{cases}$$

$$x[n] = \sin\left(\frac{2\pi}{N}mn\right) \longleftrightarrow X[k] = \begin{cases} \frac{\sqrt{N}}{2j} & k = m \\ -\frac{\sqrt{N}}{2j} & k = N-m \\ 0 & \text{else} \end{cases}$$

$$x[n] = 1 \longleftrightarrow X[k] = \sqrt{N}\,\delta[k]$$

$$\left(\vec{X} = \sqrt{N}\,\vec{\delta}_0\right)$$

δ shift

$$x[n] = e^{j\frac{2\pi}{N}mn} \longleftrightarrow X[k] = \sqrt{N}\,\delta[k-m]$$

$$\left(\vec{X} = \sqrt{N}\,\vec{\delta}_m\right)$$

shift by m

duality

$$x[n] = \delta[n] \longleftrightarrow X[k] = \frac{1}{\sqrt{N}}$$

$$x[n] = \delta[n-m] \longleftrightarrow X[k] = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}mk}$$

$$x[n] = \text{boxcar} \longleftrightarrow X[k] = \text{sinc}$$

$x[n] = \text{sinc} \longleftrightarrow X[k] = \text{boxcar}$

III. LTI Systems

$$\underset{\text{Input}}{x[n]} \longrightarrow \square \longrightarrow \underset{\text{Output}}{y[n]}$$

- Linear: additivity + scaling

  - additivity: If $x_1[n] \longrightarrow \square \longrightarrow y_1[n]$

    $x_2[n] \longrightarrow \square \longrightarrow y_2[n]$

    Then,

    $$\hat{x} = x_1 + x_2 \longrightarrow \square \longrightarrow \hat{y} = y_1 + y_2$$

  - scaling: If $x[n] \longrightarrow \square \longrightarrow y[n]$

    Then,

    $$\hat{x} = \alpha x \longrightarrow \square \longrightarrow \hat{y} = \alpha y$$

- Time Invariance

$$\hat{x}[n] = x[n-n_0] \longrightarrow \square \longrightarrow \hat{y}[n] = y[n-n_0]$$

IV. Convolution

- Definition and Calculation

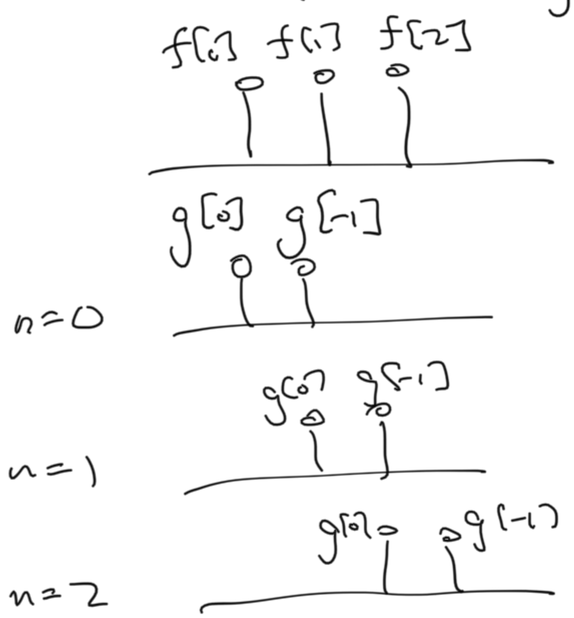
$$(f * g)[n] = \sum^{\infty} f[k]\, g[n-k]$$

$k = -\infty$

By hand calculation:

"Flip and Slide"

- like cross-correlation from 16A, except one signal has to be flipped first

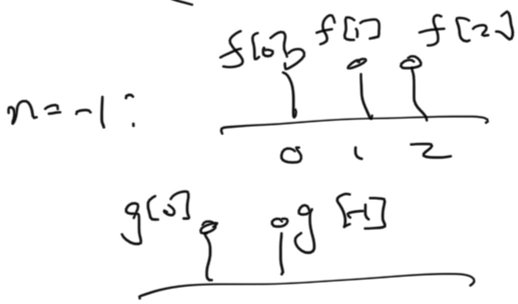


$y[0] = f[0]g[0] + f[1]g[-1]$

$y[1] = f[1]g[0] + f[2]g[-1]$

$y[2] = f[2]g[0]$

But wait there's more! Also slide to the left ($n<0$)



$y[-1] = f[0]g[-1]$

- Relation to LTI Systems

Call $\vec{h}$, the output of the system if the input is $\vec{\delta}_0$, the "impulse response"

$$x[n] = \delta[n] \longrightarrow \boxed{} \longrightarrow y[n] = h[n]$$

Any signal can be represented as a linear combination of shifted impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$\downarrow$$

$$\boxed{h}$$

$$\downarrow$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = (x * h)[n]$$

($\sum$: additivity; $x[k]$: scaling; $h[n-k]$: Time-invariance)

$$x[n] \longrightarrow \boxed{h} \longrightarrow y[n] = (x * h)[n]$$

V. Circulant Matrices

Consider a periodic DT LTI system w/ impulse response $\vec{h} = [h_0 \ \dots \ h_{N-1}]$

The output is given by a circular convolution of one period of the input w/ $h[n]$, and can be written as:

$$\vec{y} = C_h \vec{x}$$

where C_h is a circulant matrix defined by $\vec{h}$:

$$C_h = \begin{bmatrix} h_0 & h_{N-1} & \dots & h_2 & h_1 \\ h_1 & h_0 & & & h_2 \\ \vdots & \vdots & & & \vdots \\ h_{N-1} & h_{N-2} & \dots & h_1 & h_0 \end{bmatrix}$$

C_h is diagonalized by the DFT basis:

$$C_h \vec{u_k} = \left(\sqrt{N}\, H[k]\right) \vec{u_k}$$

$$C_h = F^* \begin{bmatrix} \sqrt{N} H[0] & & & 0 \\ & \sqrt{N} H[1] & & \\ & & \ddots & \\ 0 & & & \sqrt{N} H[N-1] \end{bmatrix} F$$

where $\vec{H} = F\vec{h}$ is the DFT of the impulse response $\vec{h}$ and $H[k]$ are the DFT coefficients

Apply to circular convolution eqn:

$$\vec{y} = C_h \vec{x}$$

$$\vec{y} = F^* \Lambda_H F \vec{x}$$

$$F\vec{y} = \Lambda_H F\vec{x}$$

$$\vec{Y} = \Lambda_H \vec{X}$$

$$\boxed{\vec{Y} = \sqrt{N}\vec{H} \odot \vec{X}}$$

Circular convolution in time is multiplication in frequency.

How do you account for nonperiodic systems, where $y[n] = (x * h)[n]$ ← normal convolution

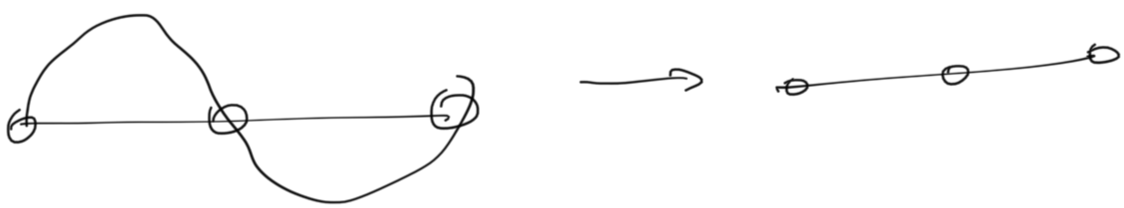
⇒ Can use circular convolution to calculate normal convolution. Just zero-pad $\vec{h}$ and $\vec{x}$, where $\vec{h}$ is length M and $\vec{x}$ is length N, to length $M + N - 1$.

VI. Sampling Theorem

If the highest frequency present in the original signal is ω_{max}, then to perfectly reconstruct?

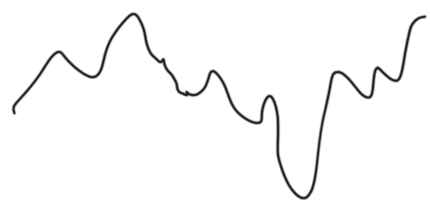
$$\omega_s > 2\omega_{max}$$

Otherwise, aliasing



HW 8, Q3: (Sampling Theorem)

$x(t) = \cos(2\pi t) + \sin(4\pi t)$



a) $\omega_{max} = 4\pi$ rad/s

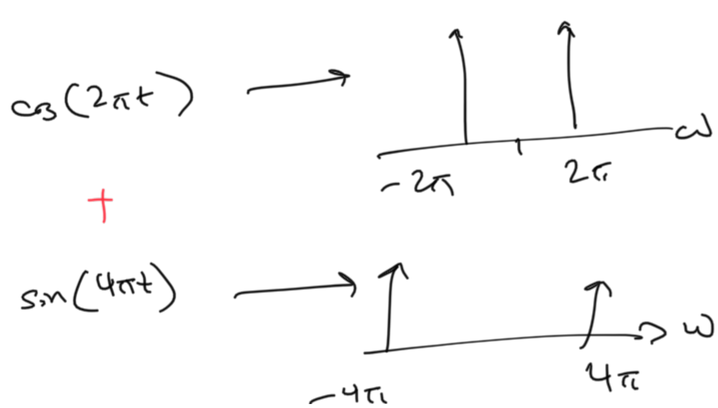
$x(t) = \cos(2\pi t) + \sin(4\pi t)$

$= \cos(\omega_1 t) + \sin(\omega_2 t)$

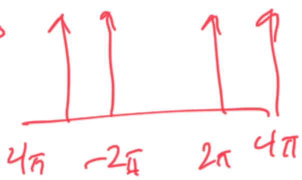
$\Rightarrow \omega_1 = 2\pi$

$\omega_2 = 4\pi$

DFT is linear



$x(t) \rightarrow$



$\omega_{max} = 4\pi$

$f = \frac{1}{T}$

$\omega = 2\pi f = \frac{2\pi}{T}$

b) If you sample every T seconds, what is ω_s?

$$\boxed{\omega_s = \frac{2\pi}{T}}$$

(If not specified sampling freq $\rightarrow$ f_s (Hz))

c) T_{min} for imperfect reconstruction?

$$\boxed{T_{min} = \frac{1}{4}\ s}$$

$\omega_s > 2\omega_{max} = 2(4\pi) = 8\pi$

$\omega_s > 8\pi$ (for perfect reconstruction)

$\omega_s \leq 8\pi$ (for imperfect reconstruction)

$\frac{2\pi}{T} \leq 8\pi \Longrightarrow T \geq \frac{1}{4}\ s$

$T_{min} = \frac{1}{4}\ s$

HW 8, Q 4 (Aliasing)

Video:  $\longrightarrow$ looks still!

30 fps (sampling at 30 fps)

a) fan is still, what could f_{real} be?

- $\frac{1}{5}$ of a rotation looks the same as 0 rotation

  $\frac{2}{5}$ of a rotation ...

  $\frac{3}{5}$ ...

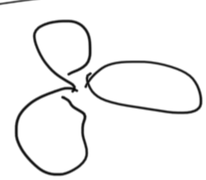
  $\Rightarrow$ $\frac{k}{5}$ rotations would look the same as no rotations

- $f_s = 30$ Hz

  $T_s = \frac{1}{f_s} = \frac{1}{30}$

- $$\boxed{f_{real} = \frac{k/5 \text{ rotations}}{T_s \text{ seconds}} = \frac{k/5}{1/30 \text{ s}} = 6k \text{ Hz}}$$

b) 3 blades

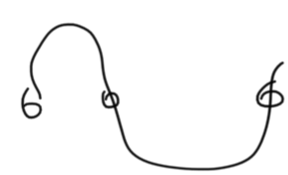


$f_{apparent} = 2$ Hz

$f_{real} = f_{apparent} + \left(\text{contribution from symmetry} \text{ — } \text{"looks still"} \right)$

$= 2 \text{ Hz} + \frac{k/3}{T_s}$

$$\boxed{f_{real} = 2 + 10k \text{ Hz}}$$

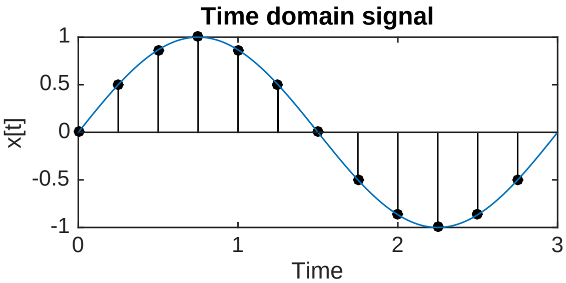


PRINT your name and student ID: ______________________________

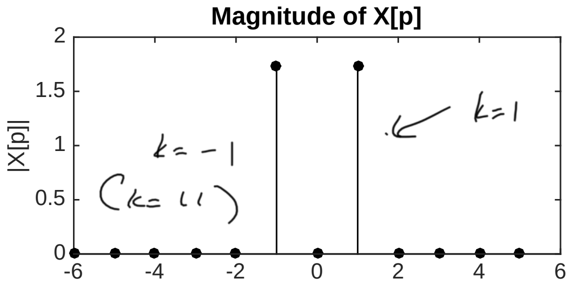
5. Matching: DFT and Sampling (12 pts)

Circle your answer. There is no need to give any justification.

(a) Given the time domain signal below with 12 samples taken over 3 seconds:

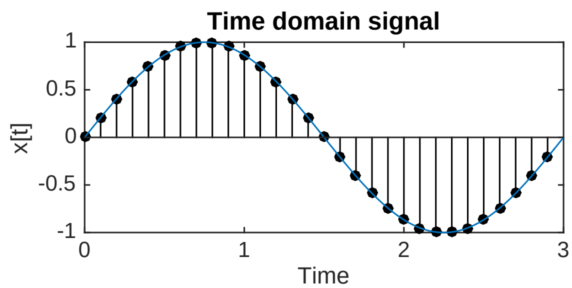


Time domain signal.

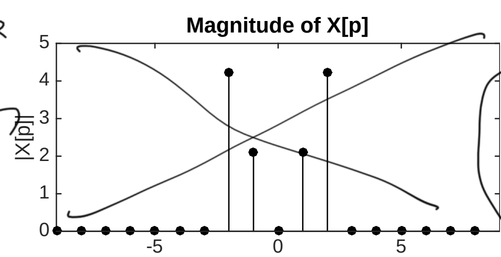


DFT magnitudes.

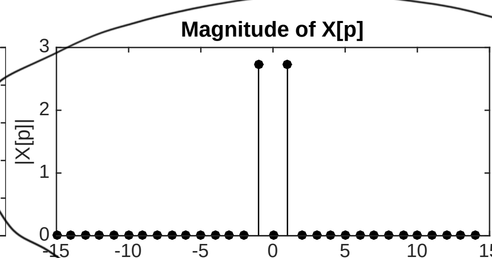
Now, we have sampled the same signal at a different sampling rate to get 30 samples over 3 seconds:



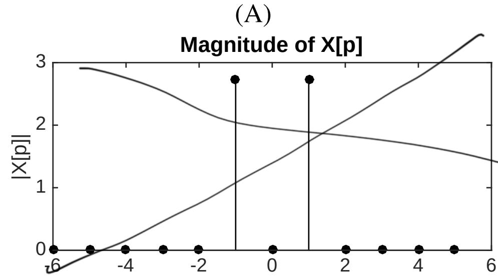
which plot below represents all the corresponding DFT coefficients?



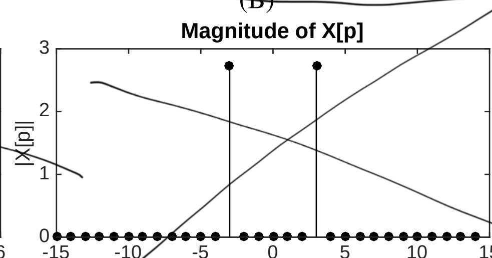
(A)



(B)



(C)



(D)

• Expect 2 spikes → still a sinusoid

• Expect magnitude $\frac{\sqrt{N}}{2} = \frac{\sqrt{30}}{2} \approx 2.7$

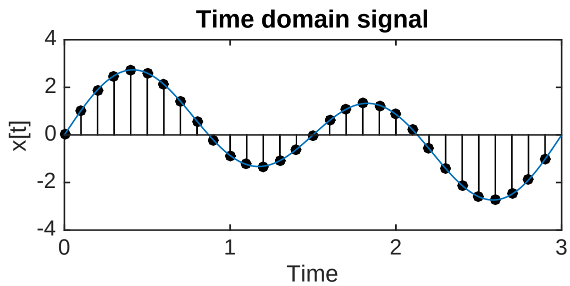
• Indices: Recall we can determine by counting how many cycles are completed → k indicates harmonic, so if you oscillate at $k\omega_0$, should complete k cycles

⟹ still one cycle, k = 1

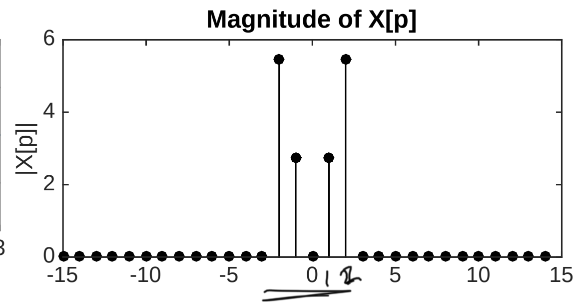
N = 30 samples

PRINT your name and student ID: ______________________________

(b) Given the time domain signal below with 30 samples taken over 3 seconds,

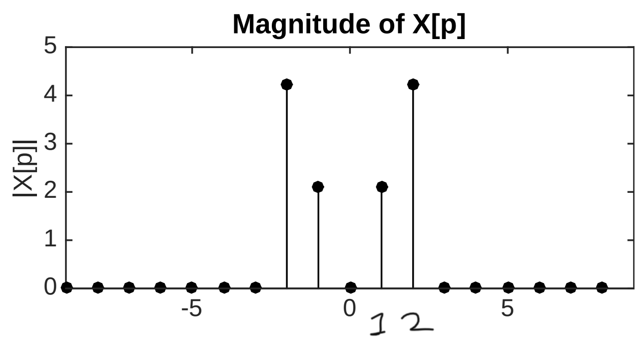


Time domain signal.

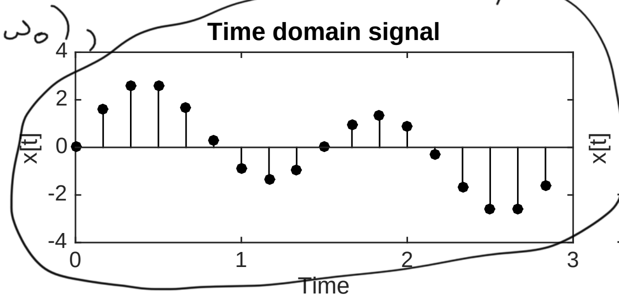


DFT magnitudes.

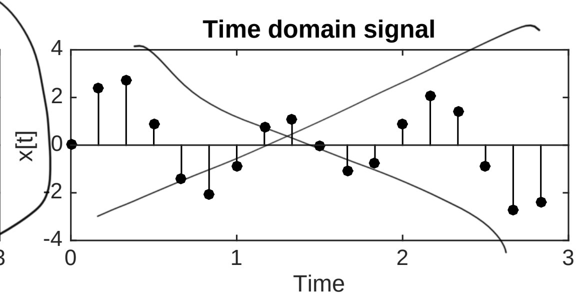
Now, we have obtained the DFT coefficients for another sampled signal, with sampling frequency of 6Hz:



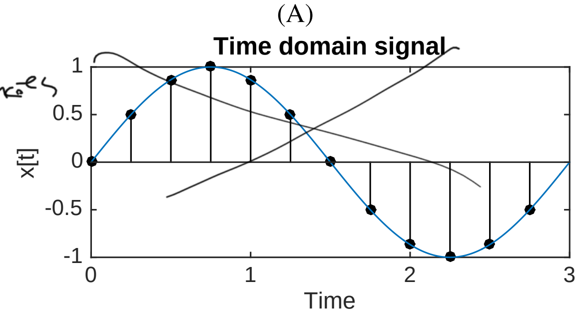
which one is the corresponding sampled signal?



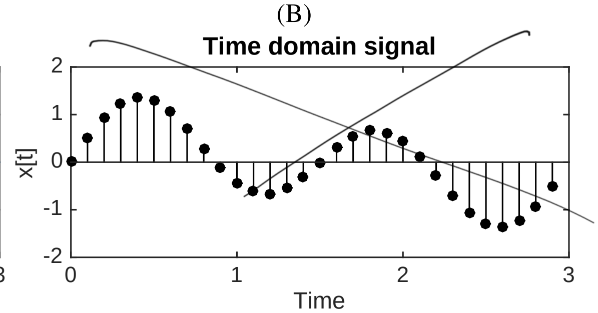
(A)



(B)



(C)



(D)

• N = 18 samples

• X[k] is nonzero at k=1,2 (−1,−2)

↓

fundamental (ω_0), 2nd harmonic ($2\omega_0$)

• same as before!

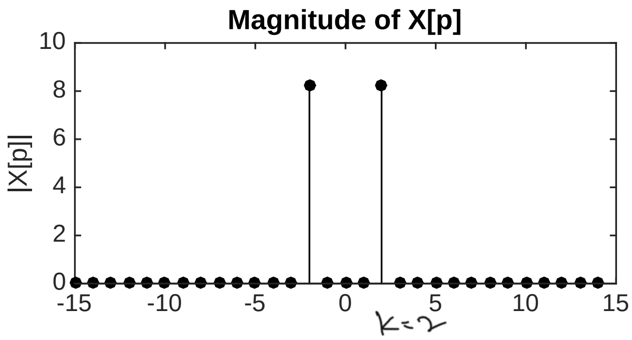
⇓

same frequencies are present! (2 freq)

A, B, C

• For A vs B, note that the relative intensities of X[1] and X[2] are unchanged ⟹ A, shape is unchanged

(c) Given the DFT coefficients of the sampled signal, obtained by taking 30 samples with a sampling frequency of 5Hz:



Which one is possibly the corresponding continuous time signal that was sampled?