Discussion 8C

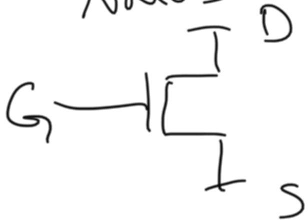
Circuits Review

- Transistors
- 1st order diff eq (RC)
- 2nd order diff eq (RLC)
- phasors
- filters
- resonance

– Problem
- Circuits Review WS Q7
- Sp20 MT1 Q6

I. Transistor Models

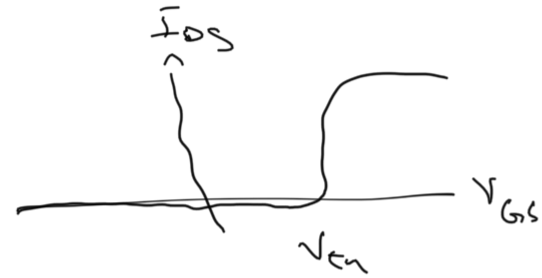
NMOS



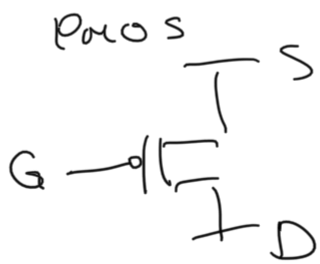
$I_{DS} > 0$

$V_{tn} > 0$

$V_{gs} \geq V_{tn}$ to close



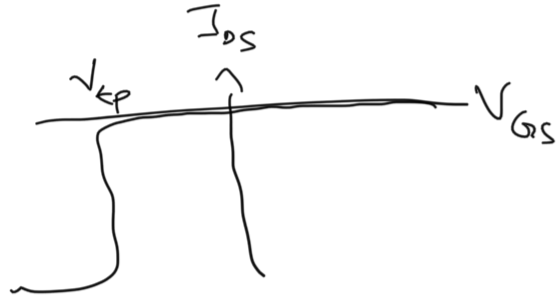
PMOS



$I_{DS} < 0$

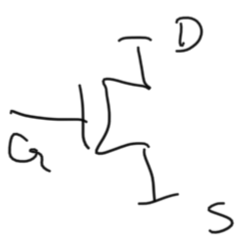
$V_{tn} < 0$

$|V_{gs}| \geq |V_{tp}|$



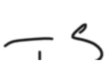
- Switch Model

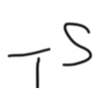
(Digital logic gates)

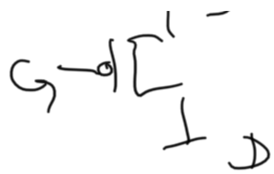


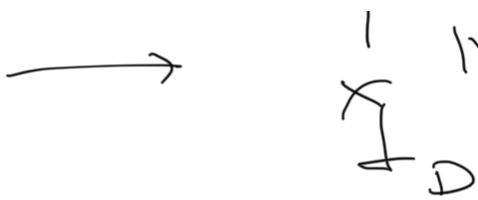
$\longrightarrow$

$V_{GS} \geq V_{tn}$ NMOS









$|V_{GS}| \geq |V_{tp}|$ PMOS

- Resistor Switch Model
  (power)

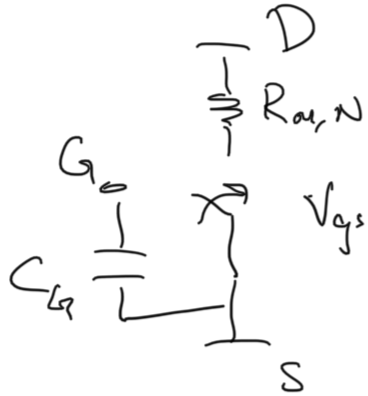
NMOS 

$V_{gs} \geq V_{tn}$

PMOS 

$|V_{gs}| \geq |V_{tp}|$

- RC Switch Model ← most important

NMOS 

$V_{gs} \geq V_{tn}$

PMOS 

$|V_{gs}| \geq |V_{tp}|$

II. 1st Order Diff Eqns

$$\frac{dx(t)}{dt} = \lambda x(t) + u(t)$$

- Homogeneous $(u(t) = 0)$

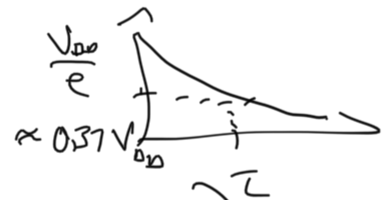
$$\frac{dx(t)}{dt} = \lambda x(t)$$

With initial condition $x(0) = x_0$

$$\Rightarrow \boxed{x(t) = x_0 e^{\lambda t}}$$

In ckt context, probably can guess something

like $\quad V_{DD}\, e^{-t/\tau} \longleftarrow \tau = RC$ (for RC ckts)

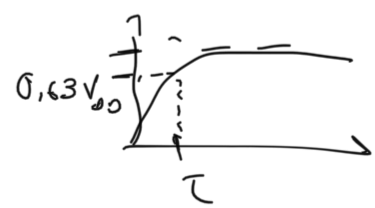
(discharging) 

- Constant Input $(u(t) = b)$

$$\frac{dx(t)}{dt} = \lambda x(t) + b$$

$$\boxed{x(t) = -\frac{b}{\lambda} + \left(x_0 + \frac{b}{\lambda}\right) e^{\lambda t} \quad \text{w/ I.C.} \quad x(0) = x_0}$$

In the context of ckts, probably can guess

something $\quad V_{DD}\left(1 - e^{-t/\tau}\right)$ 

(charging curve)

- General case

$$\frac{dx(t)}{dt} = \lambda x(t) + u(t), \quad x(t_0) = x_0$$

$$\boxed{x(t) = e^{\lambda(t - t_0)} x_0 + \int_{t_0}^{t} e^{\lambda(t - t')} u(t')\, dt'}$$

(IV.) 2nd Order Diff Eq

Turn into matrix diff eq:

Ex: series RLC, use V_c and $\frac{dV_c}{dt}$ as your state variables

Matrix diff. eq: $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

2nd order diff eq $\frac{d}{dt}\begin{bmatrix} V_c \\ \frac{dV_c}{dt} \end{bmatrix} = A \begin{bmatrix} V_c \\ \frac{dV_c}{dt} \end{bmatrix}$

- Solution

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

By diagonalization argument:

$$x_i(t) = \sum_k C_k e^{\lambda_k t}$$

Procedure

(1) Guess $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix}$.

(2) Evaluate $\vec{x}(0)$ and $\frac{d\vec{x}}{dt}(0)$ to obtain 4 eqns in 4 unknowns.

$\begin{bmatrix} \alpha_1 + \alpha_2 \end{bmatrix} \quad \begin{bmatrix} x_1(0) \end{bmatrix}$ known

$$\vec{x}(0) = \begin{bmatrix} \cdots \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} \cdots \\ x_2(0) \end{bmatrix} \quad \text{(initial conditions)}$$

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} \alpha_1 \lambda_1 + \alpha_2 \lambda_2 \\ \beta_1 \lambda_1 + \beta_2 \lambda_2 \end{bmatrix} = A \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad \text{known}$$

(3) Solve!

(IV.) Phasors

Note that a sinusoid can be written as follows:

$$V_0 \cos(\omega t + \varphi) = \mathrm{Re}\{\underbrace{V_0 e^{j\varphi}}_{\tilde{V}} e^{j\omega t}\} = \frac{\tilde{V} e^{j\omega t} + \bar{\tilde{V}} e^{-j\omega t}}{2}$$

Mathematically convenient to represent sinusoids as this complex number, the "phasor representation"

$$\boxed{\tilde{V} = V_0 e^{j\varphi} \xleftrightarrow{\text{correspond to}} V_0 \cos(\omega t + \varphi)}$$

$\tilde{V}$ encodes magnitude + phase information (only things that change in linear circuits)

$$e^{j\omega t} \longrightarrow \boxed{H} \longrightarrow \lambda e^{j\omega t}$$

transfer function

Procedure

$H(j\omega)$

① Adopt a cosine reference

$$\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

② Transform circuit to phasor domain

$$R \rightarrow Z_R = R$$

$$L \longrightarrow Z_L = j\omega L$$

$$C \longrightarrow Z_C = \frac{1}{j\omega C}$$

③ Use KCL, KVL to solve!

④ Transform back to the time domain

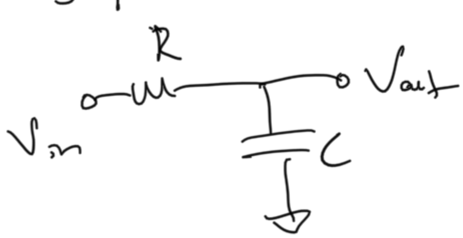
$$V_{out} \cos(\omega t + \varphi) = \mathrm{Re}\left\{ \tilde{V}_{out} e^{j\omega t} \right\}$$

$$\tilde{V}_{out} = V_{out} e^{j\varphi_{out}}$$

V. Filters

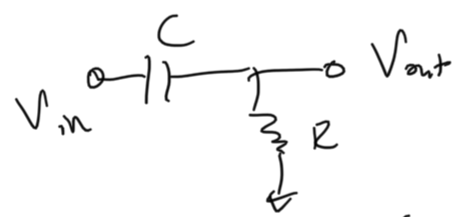
- Building Blocks

Simple RC LPF



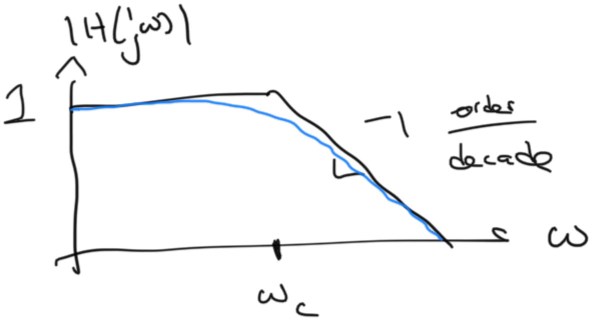
$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

Simple RC HPF

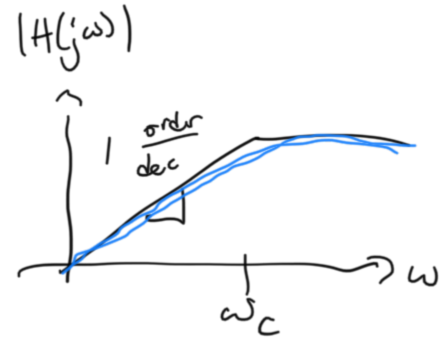


$$H(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

"Transfer functions have magnitude and phase



log-log



"

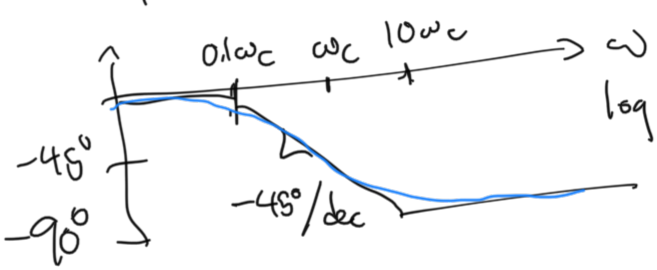
$\omega_c =$ "corner / cutoff / 3dB / pole frequency"

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} |H_{max}|$$

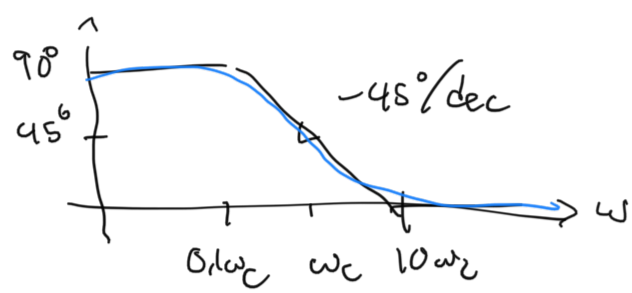
LPF $H(j\omega) = \dfrac{1}{1 + j\frac{\omega}{\omega_c}}$

HPF $H(j\omega) = \dfrac{j\frac{\omega}{\omega_c}}{1 + j\frac{\omega}{\omega_c}}$

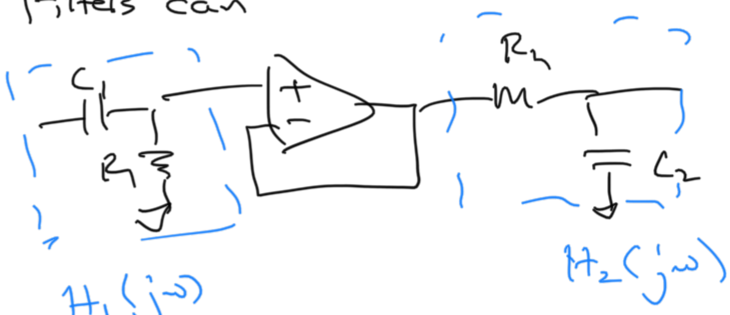
For simple RC, $\omega_c = \dfrac{1}{RC}$



log-linear



- Filters can be cascaded



$$H_{tot}(j\omega) = H_1(j\omega) H_2(j\omega)$$

B/c of log-properties,

multiplication $\longrightarrow$ add plots.

- Poles and Zeroes

$$H(j\omega) = K \frac{(j\omega)^{N_{z0}} \left(1 + j\frac{\omega}{\omega_{z1}}\right)\left(1 + j\frac{\omega}{\omega_{z2}}\right)\cdots}{(j\omega)^{N_{p0}} \left(1 + j\frac{\omega}{\omega_{p1}}\right)\left(1 + j\frac{\omega}{\omega_{p2}}\right)\cdots}$$

zeroes

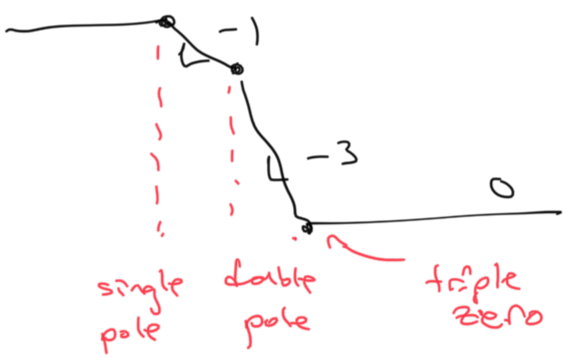
poles

In the log-log magnitude plot,

a) Each zero adds an additional slope of +1

b) Each pole adds an additional slope of −1

c) poles and zeroes can cancel



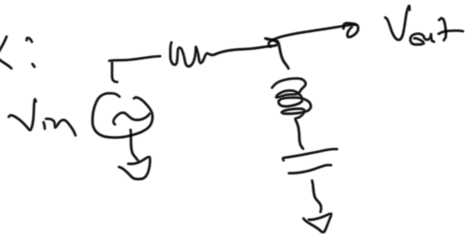
(VI.) Resonance

Impedances of L and C cancel

$j\omega C \quad j$

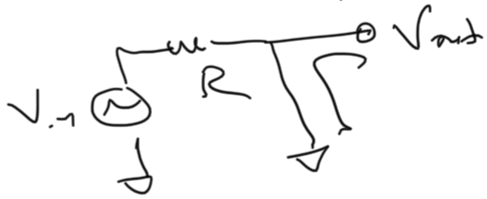
$$\Longrightarrow \frac{1}{j\omega C} + j\omega L = 0 \longrightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

EX:

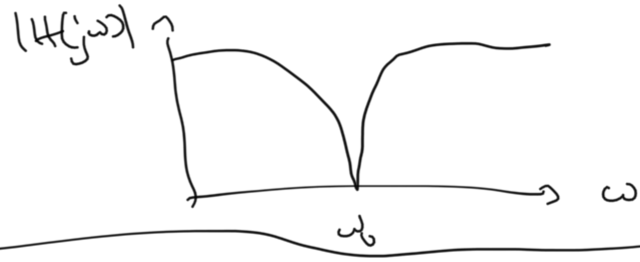


Antiresonant RLC Notch Filter

At $\omega = \omega_0$, looks like:



$\Longrightarrow V_{out} = 0$



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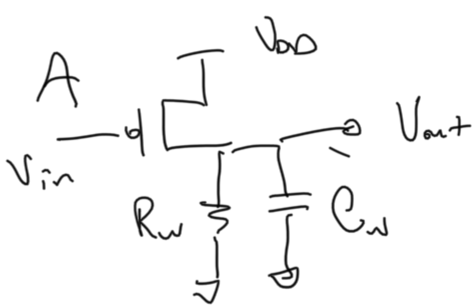
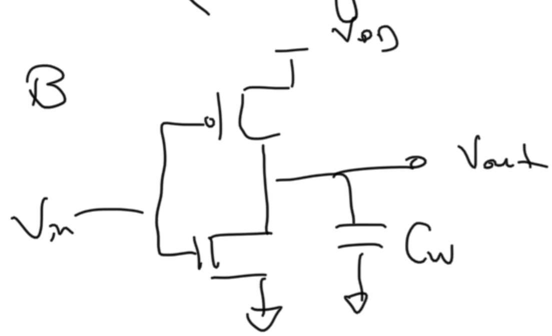
(1) Circuits Review WS, Q7 (Powering a Clock)

Ckt A



Ckt B



$R_w = 500\,\Omega$

$C_w = 3\,\mu F$

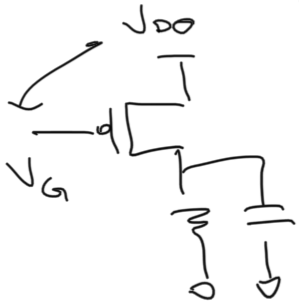
$|V_{tn}| = 0.7\,V$

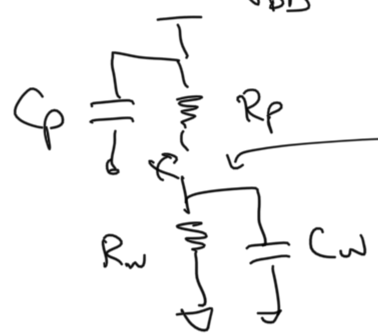
a)

Ckt A: $R_p = 1\,k\Omega$

$C_p = 5\,\mu F$

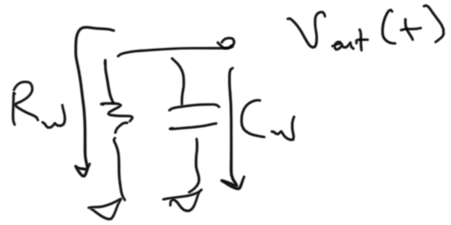
$V_{out}(0) = V_{DD}$





$|V_{gs}| \geq |V_{tp}|$

$t=0$: V_{in} switches to HIGH, PMOS off



$V_{out}(0) = V_{DD}$

Expect: $V_{out}(t) \longrightarrow$ discharging

$$V_{out}(t) = V_{DD}\, e^{-t/RC}$$

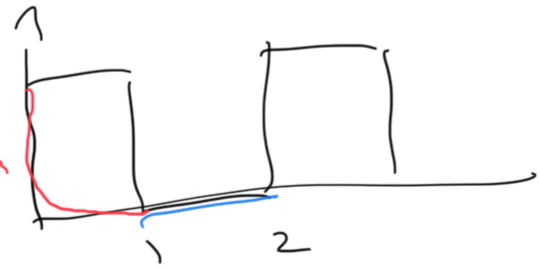
$T \in [0, 1)$

$$\frac{V_{out}}{R_W} + i_{CW} = \frac{V_{out}}{R_W} + C_W \frac{dV_{out}}{dt} = 0$$

$$\frac{dV_{out}}{dt} = \underbrace{-\frac{1}{R_W C_W}}_{\lambda} V_{out}$$

$$\boxed{\begin{aligned} V_{out}(t) &= V_{out}(0)\, e^{-t/R_w C_w} \\ \tau_{A1} &= R_w C_w = 500\,\Omega \cdot 3\,\mu F = 1.5\ ms \end{aligned}}$$

$t \in [1, 2)$

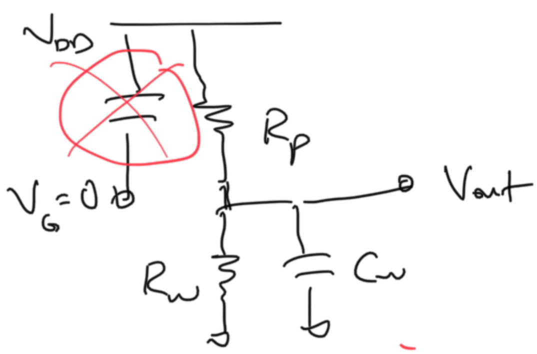


$\tau_{A1} \ll 1\,s$

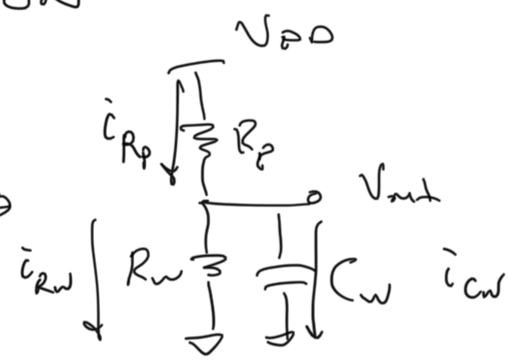
$\Downarrow$

$V_{out}(1) = 0$

V_{in} LOW, PMOS ON



$\Rightarrow$



$$\overbrace{\frac{V_{DD}-V_{out}}{R_p}}^{i_{Rp}} = \overbrace{\frac{V_{out}}{R_w}}^{i_{Rw}} + \overbrace{C_w \frac{dV_{out}}{dt}}^{i_{Cw}}$$

Expect: RC charging curve

$$\frac{dV_{out}}{dt} = \frac{V_{DD}}{R_p C_w} - \frac{V_{out}}{R_p C_w} - \frac{V_{out}}{R_w C_w}$$

$$= \frac{V_{DD}}{R_p C_w} - V_{out}\left(\frac{1}{R_p} + \frac{1}{R_w}\right)\frac{1}{C_w}$$

$$= \frac{V_{DD}}{R_p C_w} - V_{out}\left(\frac{R_p + R_w}{R_p R_w C_w}\right)$$

$$= -\left(\frac{R_p + R_w}{R_p R_w C_w}\right)\left[V_{out} - \frac{V_{DD}}{R_p C_w}\,\frac{R_p R_w C_w}{R_p + R_w}\right]$$

$$= -\left(\frac{R_p + R_w}{R_p R_w C_w}\right)\left[V_{out} - V_{DD}\,\frac{R_w}{R_p + R_w}\right]$$

$$\tilde{V}_{out} = V_{out} - V_{DD}\,\frac{R_w}{R_p + R_w}$$

$$\frac{d\tilde{V}_{out}}{dt} = \frac{dV_{out}}{dt}$$

$$\Longrightarrow \quad \frac{d\tilde{V}_{out}}{dt} = -\left(\frac{R_p + R_w}{R_p R_w C_w}\right)\tilde{V}_{out}$$

$$\tilde{V}_{out}(t) = \tilde{V}_{out}(1)\, e^{-(t-1)\left(\frac{R_p + R_w}{R_p R_w C_w}\right)}$$

$$\tau = (R_p \parallel R_w)\, C_w$$

$$V_{out} - (\quad)\, V_{DD} = \left[\underbrace{\cancel{V_{out}(1)}}_{0} - (\quad)\, V_{DD}\right] e^{-\frac{(t-1)}{\tau}}$$

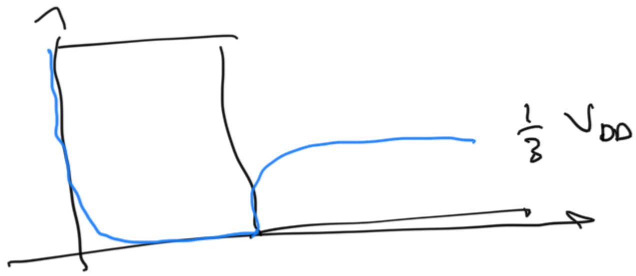
$$\boxed{V_{out} = \frac{R_w}{R_p + R_w}\, V_{DD}\left(1 - e^{-(t-1)/\tau_{A2}}\right)}$$

where $\tau_{A2} = (R_p \parallel R_w)\, C_w$

$$\tau_{A2} = \left(\frac{1}{500} + \frac{1}{1000}\right)^{-1} 3\ \mu F$$

$$= \frac{1}{3}\ k\Omega \times 3\ \mu F = 1\ ms$$

$\tau_{A2} << 1\ s$,



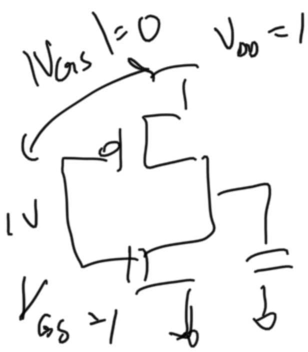
$$V_{out}(t) = \frac{R_n}{R_p + R_n} V_{DD} \left(1 - e^{-t/\tau}\right)$$

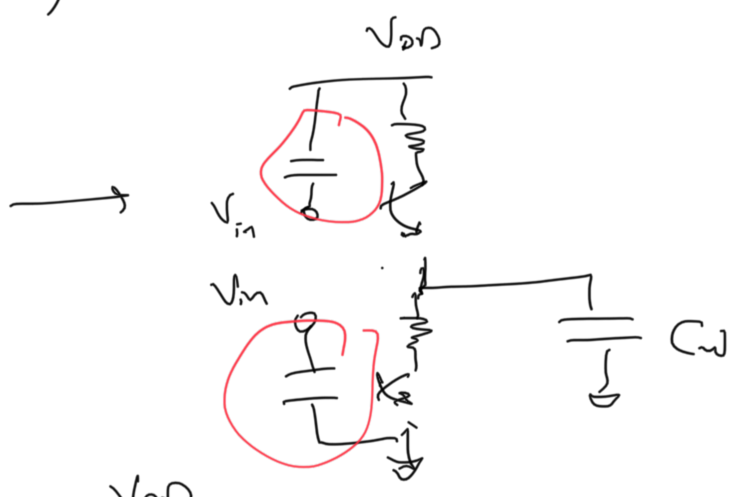
$$= \frac{1}{3} V_{DD} \left(1 - e^{-t/\tau}\right)$$

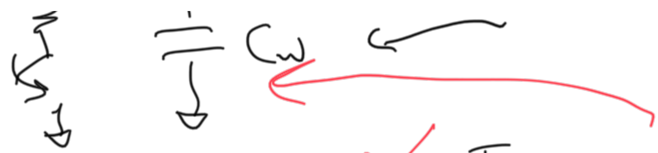
b) Ckt B

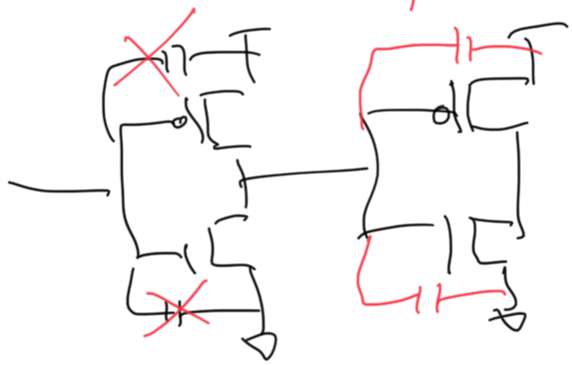
$R_p = 1\ k\Omega$, $C_p = 5\ \mu F$

$R_n = 1\ k\Omega$, $C_n = 2\ \mu F$









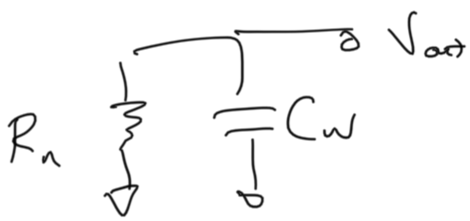
$t=0$: $V_{out}(0) = V_{DD}$

V_{in} = HIGH

$\downarrow$

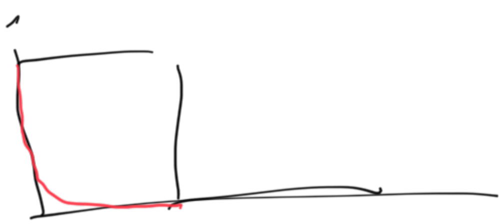
NMOS on, PMOS off

$t \in [0,1)$



$\tau_{B_1} = R_n C_W$

$$\Rightarrow \boxed{\begin{aligned} V_{out}(t) &= V_{DD}\, e^{-t/R_n C_W} \\ \tau_{B_1} &= 1\,k\Omega \cdot 3\,\mu F = 3\,ms \end{aligned}}$$



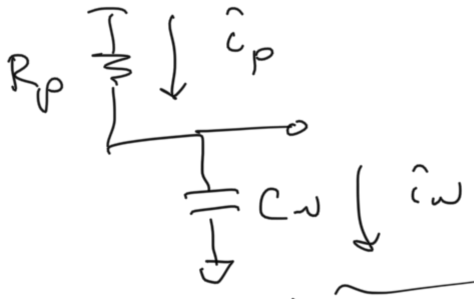
$\downarrow$

$V_{out}(1) = 0$

$t \in [1,2)$

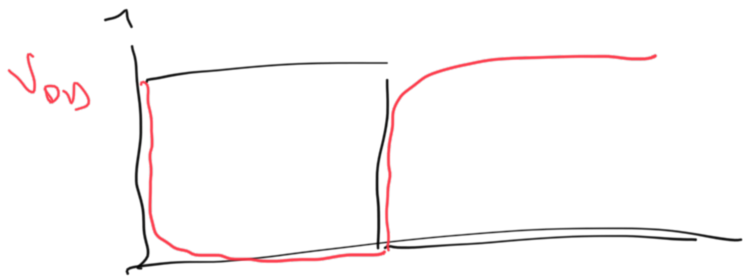
$\Longrightarrow$ PMOS on, NMOS off

$V_{in} = \text{LOW}$



$$\overset{i_p}{\frac{V_{DD} - V_{out}}{R_p}} = \overset{i_w}{C_w \frac{dV_{out}}{dt}}$$

$$\Rightarrow \boxed{\begin{gathered} V_{out}(t) = V_{DD}\left(1 - e^{-(t-1)/R_p C_w}\right) \\ \tau_{B2} = R_p C_w = 3 \text{ ms} \end{gathered}}$$



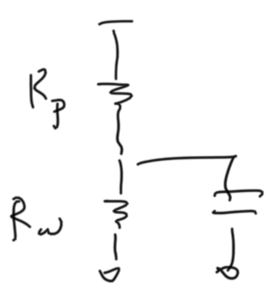
c) Energy dissipation

Model A:

[0,1)

$$\boxed{\begin{aligned} U &= \frac{1}{2} C_w V_{DD}^2 \\ &= 1.5 \times 10^{-6} V_{DD}^2 \end{aligned}}$$

[1,2)



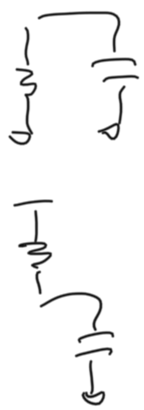
$$\underline{U = \int P(t)\,dt} = \int I(t) V_{DD}\,dt$$

charges super quickly: approximate capacitor as ... instantly being charged ... 0

$$P = \frac{V_{DD}^2}{R_p + R_w}$$

$$\boxed{U = \int_1^2 P\,dt = \frac{V_{DD}^2}{R_p + R_w} \times 1\,s = 6.33 \times 10^{-3}\, V_{DD}^2}$$

Model B:



$$\boxed{\begin{aligned} U &= \frac{1}{2} C_w V_{DD}^2 = 1.5 \times 10^{-6}\, V_{DD}^2 \\ U &= \frac{1}{2} C_w V_{DD}^2 = 1.5 \times 10^{-6}\, V_{DD}^2 \end{aligned}}$$

Clearly, model A dissipates more energy

d) Speed?

Total delay for ckt A: $\tau_{1A} + \tau_{2A} = 1.5\,ms + 1\,ms = \boxed{2.5\,ms}$

" " ckt B: $\tau_{1B} + \tau_{2B} = 3\,ms + 3\,ms = \boxed{6\,ms}$

Ckt A is faster, but at the cost of more energy dissipating