Discussion 1A Notes

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Discussion 1A notes for students taking EECS 16B during Summer Sessions 2020. Topics include: logistics, vector spaces and subspaces, eigenvalues and eigenvectors, basic circuit elements, Kirchoff's Current Law (KCL), nodal analysis, equivalent resistance, equivalent capacitance, and Thevenin/Norton equivalence.

I. LOGISTICS

A. About Me

TA: Matthew (Matt) Yeh
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Discussion: M-Th, 1-3pm PST (LOST section)
OH: Mondays, 5-6pm PST
Website: mudyeh.github.io/teaching/ee16b/su20
Slack channel: ee16bsummer2020.slack.com; join the channel #1pm-discussion, which we will be using for polling during discussion. I will also post my notes after the discussion in the channel, although since Slack has a tendency to delete files after a certain time, for posterity I will also post my notes on my website (linked above).

B. LOST Section

Note that LOST section is two hours long instead of the normal one. This way, we have more time to go over the worksheet in all the conceptual depth it deserves. I'm hoping that these sections will be heavily student-driven, i.e. your questions drive the flow of the section so that together, we can solidify conceptual foundations for anyone feeling shaky about the material. If we end up not using the full two hours for the worksheet, I plan on using the extra time for bonus content, which might take the form of extra practice problems (in-scope) or fun extensions of the material beyond 16B (out-of-scope).

C. Homework

There is a homework this week. It will be five problems plus a "find a group" problem.

Homeworks are on a Monday to Tuesday schedule, i.e. they are assigned on Monday and due on Tuesday (8 days later). See Piazza for the specific details. Self-grades are on a Wednesday to Tuesday schedule. Don't forget to do them!

Finally, Homework Party is every Friday, from 1-5pm PST. I highly recommend going to the HW Parties, as they are a great place to get help as well as meet some of your fellow students.

II. SELECTED LINEAR ALGEBRA REVIEW

A. Vector Spaces and Subspaces

Definition 1 (Vector Space). A vector space $\mathbb{V}$ over a field $\mathbb{F}$ is a set of vectors along with two operations, scalar multiplication and vector addition. They must obey the following ten properties:

- Associativity of Addition: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ for $\vec{u}, \vec{v} \in \mathbb{V}$.
- Commutativity of Addition: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{V}$.
- Identity Element of Addition: $\exists \vec{0} \in \mathbb{V}$ such that $\vec{v}+\vec{0} = \vec{v}$ for $\vec{v} \in \mathbb{V}$.
- Inverse Element of Addition: For any $\vec{v} \in \mathbb{V}$, $\exists -\vec{v} \in \mathbb{V}$ such that $\vec{v}+(-\vec{v}) = \vec{0}$.
- Closure under Vector Addition: For any two vectors $\vec{u}, \vec{v} \in \mathbb{V}$, $\vec{u}+\vec{v} \in \mathbb{V}$.
- Associativity of Scalar Multiplication: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for $\alpha, \beta \in \mathbb{F}$.
- Identity Element of Scalar Multiplication: $\exists 1 \in \mathbb{F}$ such that $1\vec{v} = \vec{v}$ for $\vec{v} \in \mathbb{V}$.
- Distributivity of Vector Addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$ for $\alpha \in \mathbb{F}$ and $\vec{u}, \vec{v} \in \mathbb{V}$.
- Distributivity of Scalar Addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v}$ for $\alpha, \beta \in \mathbb{F}$ and $\vec{v} \in \mathbb{V}$.
- Closure under Scalar Multiplication: For $\vec{v} \in \mathbb{V}$ and $\alpha \in \mathbb{F}$, $\alpha\vec{v} \in \mathbb{V}$.

Scalars are usually taken to be the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$.

Note: In section I briefly mentioned that vector spaces have ten properties, but did not list them explicitly. Here they are!

A vector space is usually described as the **span** of a set of basis vectors. Recall that the span of a set of vectors is the set of all linear combinations of those vectors.

Definition 2 (Basis). A set of vectors $\{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}\}$ is a basis for some vector space $\mathbb{V}$ if the vectors are all linearly independent and $\mathbb{V} = \text{span}\{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}\}$.

Basis sets are not unique, although they all have the same cardinality (number of elements). The cardinality of a basis for a given vector space $\mathbb{V}$ is known as the **dimension** of the vector space.

Often we are concerned with subsets of known vector spaces. These are known as **vector subspaces**.

Definition 3 (Vector Subspace). A vector subspace $\mathbb{W}$ of a vector space $\mathbb{V}$ is a subset of $\mathbb{V}$ that obeys the following three properties:

- Contains the zero vector $\vec{0}$.
- Closure under scalar multiplication.
- Closure under vector addition.

There are four fundamental matrix subspaces we are typically concerned with.

Definition 4.1 (Column Space). The column space of a matrix A, sometimes denoted as $C(A)$, is defined as the span of the columns of A. That is, if

$$\begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

then the column space is given as $C(A) = \text{span}\{\vec{a_1}, \vec{a_2}, \ldots, \vec{a_n}\}$. Note that the columns do not necessarily form a basis set, as they are not necessarily linearly independent.

Definition 4.2 (Row Space). The row space of a matrix A, sometimes denoted as $R(A)$, is defined as the span of the rows of A. Note that $R(A) = C(A^T)$ by definition of the transpose.

Definition 4.3 (Null Space). The null space of a matrix A, sometimes denoted as $N(A)$, is defined as the set of all vectors $\vec{v}$ such that $A\vec{v} = \vec{0}$.

Definition 4.4 (Left Null Space). The left null space of a matrix A, sometimes denoted as $N(A^T)$, is defined as the set of all vectors $\vec{w}$ such that $A^T\vec{w} = \vec{0}$.

The of a null space will come in handy in the next section, when trying to find eigenvalues and eigenvectors of a matrix.

B. Eigenvalues and Eigenvectors

The key equation regarding eigenvalues and eigenvectors is:

$$A\vec{v} = \lambda\vec{v}$$

For this to be possible, A must be a square matrix. Normally, multiplying a vector by a square matrix scales and rotates the vector, geometrically speaking. However, for these special vectors we call eigenvectors, the matrix only scales the vector!

Eigenvectors and eigenvalues find numerous applications in the physical sciences as well as engineering – for example, in quantum mechanics, measurable values of some observable (for example, position) are given by the eigenvalues of the Hermitian operator that represents the observable (not in scope). Throughout EE16B, you will see eigenstuff pop up over and over and over again.

To solve for the eigenvalues and eigenvectors, we wish to solve the above equation.

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= \vec{0} \\ A\vec{v} - \lambda I\vec{v} &= \vec{0} \end{aligned}$$

The problem reduces to finding the set of all nontrivial vectors $\vec{v}$ that send $A - \lambda I$ to 0! In other words, we wish to find the null space of $A - \lambda I$ for some value of λ. But which λ's will work?

It turns out that a matrix M with a nontrivial null space (i.e., more than just the $\vec{0}$ vector) will satisfy the equation

$$\det M = 0$$

where det indicates the determinant. I will not go over the specifics of calculating determinants in this note, but in the end, if you take $\det(A - \lambda I) = 0$, you should end up with a polynomial in λ known as a "characteristic polynomial." For example, consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

We go through the motions:

$$\begin{aligned} \det(A - \lambda I) = \det &\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} \\ (1-\lambda)^2 - 0 &= 0 \\ \lambda &= 1 \end{aligned}$$

In this case, it turns out we have a repeated eigenvalue of $\lambda = 1$. In general, a nonrepeated eigenvalue will have one associated eigenvector. However, repeated ones will have up to the multiplicity of the eigenvalue. In the case that the number of eigenvectors is less than the multiplicity, we have a nondiagonalizable matrix.

Now we solve for the eigenvectors, which in this case amounts to finding the null space of $A - I$.

$$\begin{aligned} (A - I)\vec{v} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0} \\ v_1 &= \text{free variable} \\ v_2 &= 0 \end{aligned}$$

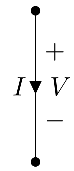
Thus any multiple of $\vec{v} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is an eigenvector of A with eigenvalue 1. A is nondiagonalizable!

III. SELECTED CIRCUITS REVIEW

A. Basic Circuit Elements

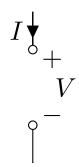
In circuit analysis, we are primarily concerned with currents and voltages. Thus, we identify each of the following circuit elements with its **I-V relation**.

1. Wire



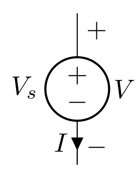
The IV relation is $V = 0, I =$ anything, i.e. a vertical line at $V = 0$ on an I-V graph. In other words, no voltage is dropped across a wire.

2. Open Circuit



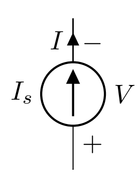
The IV relation is $V =$ anything, $I = 0$, i.e. a horizontal line at $I = 0$ on an I-V graph. In other words, no current passes through an open circuit.

3. Voltage Source



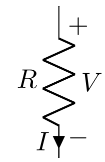
The IV relation is $V = V_s, I =$ anything, i.e. a vertical line at $V = V_s$ on an I-V graph. In other words, an ideal voltage source will pass whatever current it takes to maintain a voltage drop of V_s across its terminals.

4. Current Source



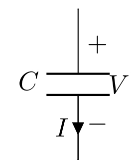
The IV relation is $V =$ anything, $I = I_s$, i.e. a horizontal line at $I = I_s$ on an I-V graph. In other words, an ideal current source will drop whatever voltage it takes to maintain a current of V_s across its terminals.

5. Resistor



The IV relation is $V = IR$, i.e. a line with slope $m = 1/R$ on an I-V graph. A short circuit can be seen as the limit as a resistor goes to 0, and an open circuit as the limit as a resistor goes to ∞.

6. Capacitor



The IV relation is $I = C\frac{dV_C}{dt}$, where V_C is the voltage across the capacitor and C is a constant known as the capacitance. Specifically, $C = \frac{Q}{V_C}$, where Q is the charge on the arbitrarily labelled "positive" terminal, and V_C is again the voltage across the capacitor. The energy stored on a capacitor is given by $U = \frac{1}{2}CV_C^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV_C$.

Throughout this discussion, we have followed **passive sign convention**. According to passive sign convention, positive current flows into the positive terminal and exits out the negative terminal. This is akin to a river flowing down a mountain, from high (gravitational) potential to low (gravitational) potential.

The benefit of passive convention is that the sign of **power** has a well-defined meaning. If $P > 0$, then the element is dissipative. If $P < 0$, then the element is a source and adds power into the circuit.

As a final note, in general the power consumed by an element is given by $P = IV$. For a resistor, we can use Ohm's law to write two alternate expressions, $P = \frac{V^2}{R} = I^2R$.

B. KCL and Nodal Analysis

Kirchoff's Current Law (KCL) follows from conservation of charge: at a junction,

$$I_{in} = I_{out}.$$

The nodal analysis method essentially uses KCL to solve a circuit.

Procedure 1 (Nodal Analysis).

1. Pick a ground (reference) node.

2. Label all remaining nodes in the circuit.

3. Label nodes and currents through each element. Add (+) and (-) signs according to passive sign convention.

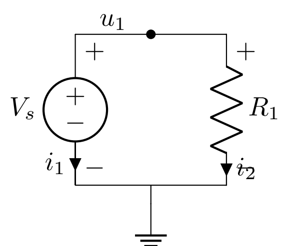
4. Write KCL equations at each node.

5. Use I-V relations to write currents in terms of circuit elements (e.g. resistances, voltage sources, etc).

6. Solve the system of equations using your favorite method.

We now go through two examples. The first one will illustrate passive sign convention, and the second will be the famed voltage divider circuit.

Example 1



Let $V_s = 5\,\mathrm{V}$ and $R_1 = 1\,\Omega$. Use nodal analysis to solve for the power dissipated or supplied by each element.

From the way we have labeled the polarities in the figure, by passive sign convention both currents must be going down. From KCL,

$$i_1 + i_2 = 0 \Rightarrow i_1 = -i_2$$

Because the voltage source is directly connected between node u_1 and ground,

$$u_1 = V_s = 5\,\mathrm{V}$$

Thus,

$$i_2 = -i_1 = 5\,\mathrm{V}/1\,\Omega = 5\,\mathrm{A}$$

For the resistor,

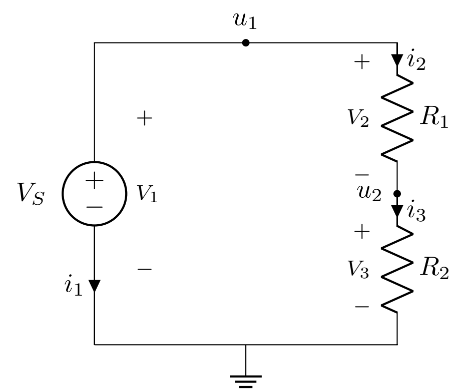
$$P = IV = 25\,\mathrm{W}$$

For the voltage source,

$$P = IV = -25\,\mathrm{W}$$

As expected, the power of the voltage source is negative, indicating that it is supplying power into the circuit whereas the resistor is dissipating power. Moreover, the sum of the powers is 0; this is from conservation of energy.

Example 2 (Voltage Divider)



Find an expression for $V_{mid} = u_2$ in terms of V_s, R_1, R_2. KCL tells us:

$$i_1 + i_2 = 0$$
$$i_2 = i_3$$

In terms of node potentials and resistances:

$$\frac{u_1 - u_2}{R_1} = \frac{u_2 - 0}{R_2}$$
$$u_1 = V_s$$
$$u_2 = V_{mid}$$

We get the potentially familiar expression,

$$V_{mid} = V_S \frac{R_2}{R_1 + R_2}$$

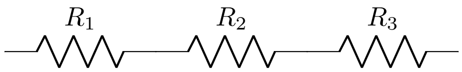
C. Equivalence I: Series and Parallel Circuits

We say two circuits are **equivalent** if they have the same I-V relation. Often we encounter circuit elements that are said to be connected in **series** or in **parallel**. It would be immensely helpful for simplifying circuit analysis if we had some way to collapse these configurations into a single equivalent element. In this section, we will focus on resistors and capacitors.

Definition 5 (Series Circuit). Circuit components are said to be connected in series if they are connected along a single uninterrupted path. By KCL, the same current flows through all elements in the series connection.

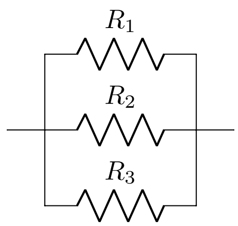
Definition 6 (Parallel Circuit). Circuit components are said to be connected in parallel if each component is connected between the same set of nodes. For two-terminal elements such as resistors, current enters at the same node, splits between each branch, and then recombines out the other node. By KVL, the same voltage drop occurs across each element in the parallel connection.

1. Series Resistors



$$R_{eq} = \sum_i R_i$$
$$= R_1 + R_2 + \dots$$

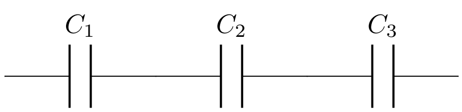
2. Parallel Resistors



$$R_{eq} = (\sum_i \frac{1}{R_i})^{-1}$$
$$= (\frac{1}{R_1} + \frac{1}{R_2} + \dots)^{-1}$$

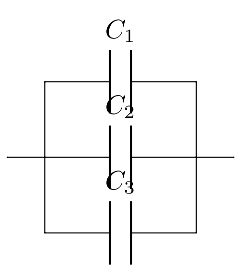
For the case of two resistors, we define a "parallel operator" $||$, where $R_1||R_2 = \frac{R_1 R_2}{R_1+R_2}$. You should check for yourself that this expression is the same as what you would get from the gross "inverse of a sum of inverses" formula.

3. Series Capacitors



$$C_{eq} = (\sum_i \frac{1}{C_i})^{-1}$$
$$= (\frac{1}{C_1} + \frac{1}{C_2} + \dots)^{-1}$$

4. Parallel Capacitors



$$C_{eq} = \sum_i C_i$$
$$= C_1 + C_2 + \dots$$

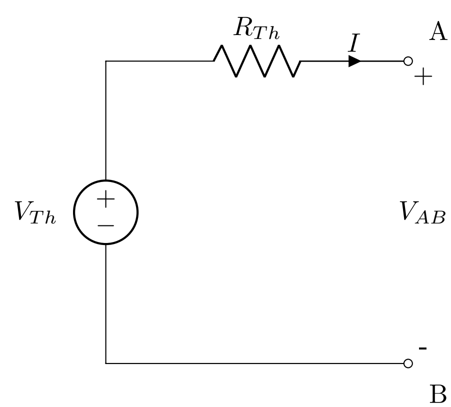
We can build some physical intuition here for why these equations are as such. Consider the resistor equation $R = \frac{\rho L}{A}$, where ρ is the resistivity, L is the length, and A is the cross-sectional area. If the resistors all have the same dimensions, stacking a bunch of resistors in series is like extending the length of one resistor – we expect the resistance to increase. Correspondingly, stacking a bunch of resistors in parallel is like widening one resistor, increasing A – we expect the resistance to decrease.

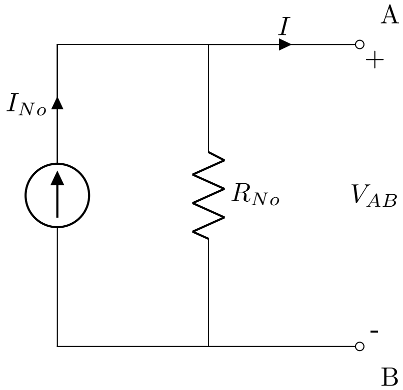
Similar intuition can be applied to the case of capacitors, although in this case, the formulas have done a bit of a switcheroo. Consider the parallel plate capacitor equation $C = \frac{\epsilon A}{d}$, where ϵ is the permittivity, A is the plate area, and d is the separation distance of the plates. Stacking a bunch of capacitors in parallel is like increasing the plate area of one capacitor, so we expect more charge to be deposited per unit voltage – the capacitance increases. Likewise, stacking a bunch of capacitors in series can be seen roughly as increasing the separation distance – we expect the capacitance to decrease.

D. Equivalence II: Thevenin and Norton Equivalent Circuits

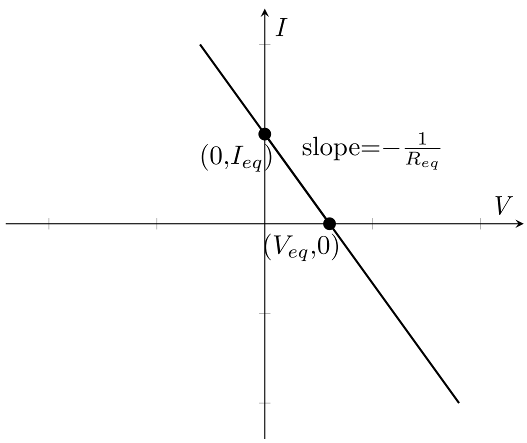
Consider a linear circuit made up of only ideal voltage sources, ideal current sources, and resistors. Suppose we want to attach a load to some set of output terminals. It would be nice if we could blackbox away the rest of the circuit so that we don't have to redo our complicated analysis (KCL, KVL) each time we attach a different load. This is the power of Thevenin's theorem and its dual, Norton's Theorem.

Essentially, it can be shown that for such a linear circuit, the voltage is given by $V = V_{eq} - R_{eq} I$. This can take the form of the Thevenin equivalent circuit using a voltage source, or the Norton equivalent circuit using a current source.





The I-V curve is given below:



Notice that the V-intercept is given by $(V_{eq}, 0)$. What circuit element has 0 current flowing through it? An open circuit!

Likewise, the I-intercept is given by $(0, I_{eq} = V_{eq}/R_{eq})$. What circuit element has no voltage drop? A short circuit!

We use this graph to conclude that:

$$V_{eq} = V_{th} = V_{oc}$$
$$I_{eq} = I_{no} = I_{SC}$$
$$R_{eq} = R_{th} = R_{no} = \frac{V_{th}}{I_{no}}$$

Procedure 2 (Thevenin and Norton Equivalents)

1. To find V_{th}, connect an open circuit across the terminals of interest. Measure out $V_{out} = V_{th}$.

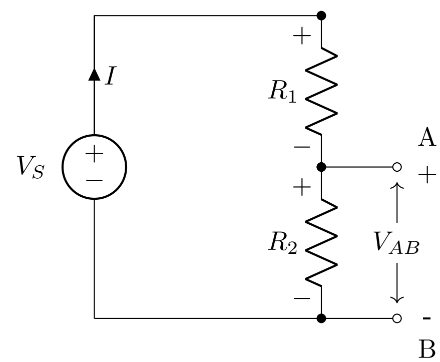
2. To find I_{no}, connect a short circuit across the terminals of interest. Measure out $I_{out} = I_{no}$.

3. To find $R_{th} = R_{no}$:

   (a) If you were able to find I_{no} and V_{th}, you are done! $R_{th} = R_{no} = \frac{V_{th}}{I_{no}}$. It is worth noting that it is not always obvious how to extract both I_{no} and V_{th}. Moreover, to create an equivalent circuit you really only need one of $\{I_{eq}, V_{eq}\}$, plus the resistance R_{eq}. Thus, in general you should use the next method.

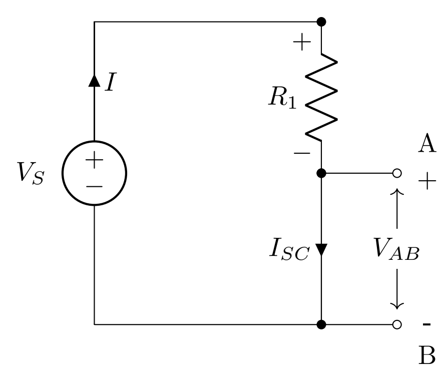
   (b) $V_{test} - I_{test}$ *Method:* "Zero out" independent sources (voltage source → wire, current source → open circuit). Either apply a test voltage V_{test} into the terminal and measure out I_{test}, or vice versa. Take the ratio $R_{eq} = \frac{V_{test}}{I_{test}}$. The intuition behind why this method works is that you are "seeing" the resistance a load would see at the output.

Example 3 (Voltage Divider)



1. $V_{th} = V_{oc}$. In this case, we already know what V_{oc} is – it's just given by the voltage divider formula! $V_{th} = V_S\frac{R_2}{R_1+R_2}$.

2. $I_{no} = I_{sc}$. If we short terminals A and B, we have a resistor in parallel with a wire, which is equivalently just a wire. You can think of this with a "path of least resistance" argument, where current will only traverse the resistance-less wire. More formally, the short circuit demands that the voltage across the resistor is 0, and thus the current across the resistor must also be 0 by Ohm's law. The circuit becomes:



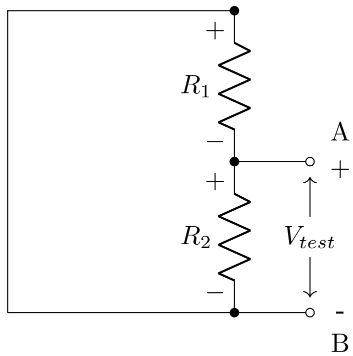
   From KCL and Ohm's law, we can argue that $I_{no} = \frac{V_s}{R_1}$.

3. In this case, we can use both techniques for finding the equivalent resistance!

(a) $R_{eq} = \frac{V_{th}}{I_{no}}$

$$R_{eq} = V_S \frac{R_2}{R_1 + R_2} / \frac{V_s}{R_1}$$
$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

(b) $V_{test} - I_{test}$
To zero out the voltage source, we replace it with a wire. The circuit becomes:



We hook up a voltage source V_{test} across the terminals A and B. We could measure out I_{test}, but in this case, we can already see the equivalent resistance without any more math!

If we "reshape" the circuit by moving around the wires connected to R_1, we might see that these resistors are in fact connected in parallel. Thus, the equivalent resistance must be $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$, exactly the same as we calculated with the other method! However, as we just saw, using the $V_{test} - I_{test}$ method can be more instructive with regards to understanding why we got the result we did.