Discussion 2A Notes

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Discussion 2A notes for students taking EECS 16B during Summer Sessions 2020. Topics include: logistics, complex numbers (Cartesian and polar form, Euler's formula, complex algebra, complex conjugate, helpful identities).

I. LOGISTICS

Homework 1: Due Tuesday, 6/30.
Homework 2: Will be assigned soon. Due next Tuesday, 7/7.
OH: Mondays, 5-6pm PST.
Homework Party: Fridays, 1-5pm PST. Reminder that these exist, and are a great way to make progress on the homework! Last week HW Party was pretty sparse :(

II. COMPLEX NUMBERS

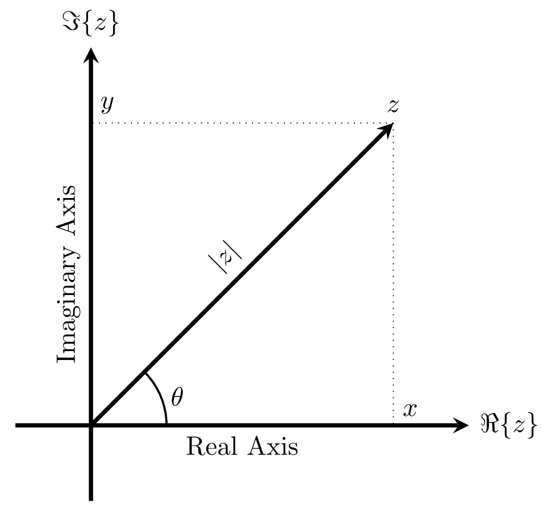
A. Introduction

A complex number z is one that can be written in the following form:

$$z = x + jy$$

where x and y are both real numbers, and j is the imaginary unit $j = \sqrt{-1}$. We use j instead of i because in electrical engineering, i is commonly used to represent current and thus we would like to avoid any potential confusion.

Since z is determined by the values of x and y, we can represent z as an ordered pair (x, y). This notation is suggestive of a two-dimensional *Cartesian* coordinate system, and thus we often visualize complex numbers as points in the complex plane, where one axis corresponds to the real number line and the other axis corresponds to the imaginary number line. See the figure below:



One might wonder if Cartesian coordinates are the only way to represent complex numbers. After all, if 16B has taught us anything, it's that there are many ways to look at the same problem. As we will expand upon in the next section, it turns out the answer is a resounding "No!"

B. Polar Coordinates

The above figure is suggestive of a different representation of complex numbers. Specifically, let us draw an arrow from the origin to the point (x, y) in the complex plane. We define the *magnitude* $|z|$ as the length of this arrow, and the *phase* θ as the angle from the positive real axis to the arrow (with positive θ going counterclockwise from the real axis). Then we may determine a complex number with the ordered pair $(|z|, \theta)$, as part of a *polar* coordinate system.

We can also convert between polar coordinates and Cartesian coordinates using some trigonometry.

1. Polar $(|z|, \theta) \Rightarrow$ Cartesian (x, y)

$$x = |z|\cos(\theta)$$
$$y = |z|\sin(\theta)$$

2. Cartesian $(x, y) \Rightarrow$ Polar $(|z|, \theta)$

$$|z| = \sqrt{x^2 + y^2}$$
$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \text{atan2}(y, x)$$

atan2(y,x) is the two-argument arctangent. We are careful to use this instead of the normal one-argument arctangent because it removes any potential ambiguity about which quadrant we are looking at. As a quick example, consider two points $A = (1,1)$ and $B = (-1,-1)$. A is located at an angle of 45° whereas B is at an angle of 225°. If we take the arctangent for point A, we obtain:

$$\theta_A = \arctan(\frac{1}{1}) = \arctan(1) = 45^\circ$$

This is all fine and dandy, but if we take the arctangent for point B, we get

$$\theta_B = \arctan(\frac{-1}{-1}) = \arctan(1) = 45^\circ$$

which is clearly not equal to 225°. By requiring two argument, atan2 can get around this situation.

In practice, if your calculator only has a arctan function, what I do to calculate the phase θ is I sketch the complex number on the complex plane and use normal arctangent. Based off my sketch, I then adjust by 180° if needed. In general, plotting is very helpful for building intuition about complex numbers!

C. Euler's Formula

In polar form, $(|z|, \theta)$ corresponds to

$$z = |z|(\cos(\theta) + j\sin(\theta))$$

It turns out polar coordinates can be wrapped into a much neater mathematical bundle than the above expression, using the famed **Euler's formula**:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Thus,

$$z = x + jy = |z|e^{j\theta}$$

To prove Euler's formula, we employ Taylor expansions (as many proofs do). Specifically, consider the Taylor expansions of e^x, $\cos(x)$, and $\sin(x)$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

If we replace x with jx, then e^{jx} is:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{j^n x^n}{n!} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} \dots \\ &= 1 - (\frac{x^2}{2!} + \frac{x^4}{4!} - \dots) + j(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + j\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ &= \cos(x) + j\sin(x) \end{aligned}$$

As I said in discussion, Euler's formula is a magical, amazing formula that also happens to pop up all the time in engineering and the physical sciences. And as a bit of trivia, plug in $x = \pi$:

$$e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1$$

Rearranging, this becomes the remarkable Euler's Identity

$$e^{j\pi} + 1 = 0$$

which has 5 fundamental constants of mathematics (the additive identity 0, the multiplicative identity 1, the natural logarithm base e, the imaginary unit j, and of course pi), as well as 3 arithmetic operations (addition, multiplication, and exponentiation) all bundled into one compact equation. People have even conducted brain imaging studies on what equation mathematicians find the most beautiful, with Euler's Identity taking first place. Ramanujan's infinite series for $\frac{1}{pi}$ did not fare so well unfortunately.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Compare for yourself!

D. Complex Algebra

Here's a list of how to do arithmetic operations with complex numbers. In general, note that we use whichever representation is more convenient for the task at hand.

Let $z_1 = |z_1|e^{j\theta_1} = x_1 + jy_1$ and $z_2 = |z_2|e^{j\theta_2} = x_2 + jy_2$:

- $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
- $z_1 z_2 = |z_1||z_2|e^{j(\theta_1+\theta_2)}$
- $z_1/z_2 = (|z_1|/|z_2|)e^{j(\theta_1-\theta_2)}$
- $z^n = |z|^n e^{jn\theta}$

Again, having both a Cartesian and polar representation is useful is because it gives us the ability to use whichever representation makes solving the problem easier. Consider exponentiation: if we were to do that operation in Cartesian form, then we would have to use an ugly binomial expansion.

$$z^n = (x + jy)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (jy)^{n-k}$$

The polar alternative is much cleaner.

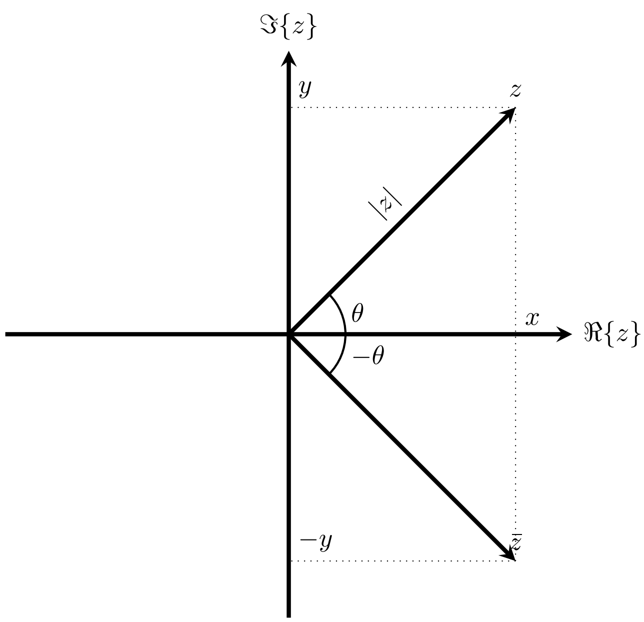
E. Complex Conjugate

Finally, we introduce another operation on complex numbers, the **complex conjugate** (represented by an overhead bar). If $z = x + jy = |z|e^{j\theta}$, then the complex conjugate of z is given by

$$\bar{z} = x - jy = |z|e^{-j\theta}$$

Perhaps unsurprisingly, the effect of a complex conjugate on the polar form can be proved using Euler's formula (it's a homework problem!).

While the definition of a complex conjugate is fairly straightforward, it does seem a bit abstract. To give a bit more geometric intuition to it, we once again plot on the complex plane.



We see that the effect of the complex conjugate is actually to reflect the complex number across the real axis.

Finally, some selected and potentially useful identities involving the complex conjugate:

- $|z|^2 = z\bar{z}$

$$\begin{aligned} z\bar{z} &= (x + jy)(x - jy) \\ &= x^2 - (jy)^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

- $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{z_1 / z_2} = \bar{z}_1 / \bar{z}_2$
- $z + \bar{z} = 2\Re[z]$ ($\Re[z]$ takes the real part of z)

$$\begin{aligned} z + \bar{z} &= (x + jy) + (x - jy) \\ &= 2x \\ &= 2\Re[z] \end{aligned}$$

- $z - \bar{z} = 2j\Im[z]$ ($\Im[z]$ takes the imaginary part of z)

$$\begin{aligned} z - \bar{z} &= (x + jy) - (x - jy) \\ &= 2jy \\ &= 2j\Im[z] \end{aligned}$$

It is worth noting that you can use the previous two identities to write cosine and sine in terms of complex exponentials. Specifically, note that

$$\cos(\theta) = \Re[e^{j\theta}] \qquad \sin(\theta) = \Im[e^{j\theta}]$$

Then by the previous two identities,

$$2\cos(\theta) = e^{j\theta} + e^{-j\theta} \qquad 2j\sin(\theta) = e^{j\theta} - e^{-j\theta}$$

Or, rearranging:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

A final property of the complex conjugate that is worth taking note of is the following:

$$z = \bar{z} \iff z \in \mathbb{R}$$

This is a useful property in proofs (for example, to show that the eigenvalues of symmetric matrices are purely real). To prove the forward direction, we use Cartesian form:

$$x + jy = x - jy$$

Equating the real and imaginary parts

$$x = x \qquad jy = -jy$$

The second equation reduces to

$$y = -y$$

which is only true if $y = 0$. Since y corresponds to the imaginary part of z, then z must be purely real.

To prove the other direction, consider $z \in \mathbb{R}$. Then z can be written as

$$z = x + 0y$$

where x is real, as usual. Taking the complex conjugate,

$$\bar{z} = x - 0y = x = z$$

Thus we have proved both directions.